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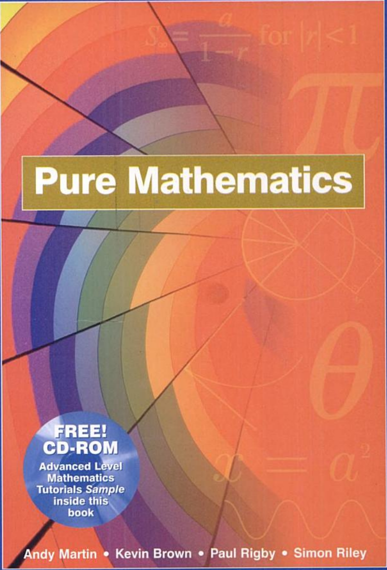
Pure Mathematics



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Andy Martin • Kevin Brown • Paul Rigby • Simon Riley

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Contents

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Introduction

Complete Advanced Level Mathematics is an exciting new series of mathematics books, Teacher Resource Files and other support materials (see page viii) from **Stanley Thornes** for those studying at Advanced Level. It has been developed following an extensive period of research and consultation with a wide number of teachers, students and others. All the authors are experienced and practising teachers and, in some cases, Advanced Level Mathematics Examiners. Chapters have been trialed in schools and colleges. All the requirements for complete success in Advanced Level Mathematics are provided by this series.

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- **What you need to know** sections covering prerequisite knowledge for a chapter.
- **Review** sections with practice questions on what you need to know.
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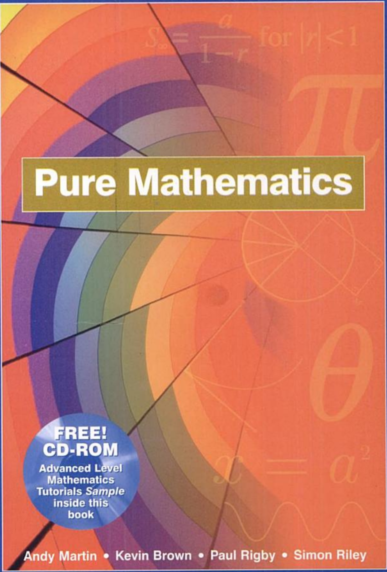

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1 Algebra I

What you need to know

- How to use index form.
- How to evaluate $\sqrt{a} \times \sqrt{a}$.
- How to collect like terms.
- How to expand brackets.
- How to solve linear equations.
- How to factorise algebraic expressions.
- How to use a division algorithm (method) without a calculator.

Review

1 Write the following in index form:

a	$x \times x$	**d**	$a \times a \times b \times b \times b$
b	$7 \times 7 \times 7$	**e**	$2x \times 2x$
c	$x \times x \times x \times x$	**f**	$3a \times 2a \times 2a$

2 Find the exact value of the following:

a	$\sqrt{36} \times \sqrt{36}$	**d**	$\sqrt{16} \times \sqrt{16}$
b	$\sqrt{4} \times \sqrt{4}$	**e**	$\sqrt{a} \times \sqrt{a}$
c	$\sqrt{9} \times \sqrt{9}$	**f**	$\sqrt{x} \times \sqrt{x}$

3 By collecting like terms simplify the following expressions:

a	$2x + 3x$	**d**	$4xy + 3yx$
b	$9x^2 + 3x^2 - 2x^2$	**e**	$2x^2 - 3x + 4$
c	$6y - 3y + 4x$	**f**	$2x + 9x - 3x$

4 Multiply out, or expand, the following terms in brackets, simplifying the answer where possible:

a	$3(x + 2)$	**g**	$(x + 1)(x + 2)$
b	$2(x - 3)$	**h**	$(x - 2)(x + 3)$
c	$-2(x + 2)$	**i**	$(x + 4)(x - 1)$
d	$-3(x - 3)$	**j**	$(x + 3)(x + 5)$
e	$-5(x + 2y - 3)$	**k**	$(x - 2)(x - 3)$
f	$6(2x - 2y - 12)$	**l**	$(x - 4)^2$

Irrationals and surds

There are some numbers that cannot be written in this form. These numbers have a decimal expansion that doesn't terminate, but goes on forever without repeating. These numbers are known as **irrational numbers**. Some examples of irrational numbers are π, $\sqrt{2}$, $\sqrt{5}$ and $\sqrt{7}$. They cannot be evaluated exactly; calculators simply give an approximation to 8, 10 or 12 decimal places. The ancient Greeks called numbers like $\sqrt{2}$, $\sqrt{5}$ and $\sqrt{7}$ **incommensurables**. Now they are often referred to as **surds**. Not all irrationals are surds; π is not a surd.

They were called incommensurables because although they could be constructed with a ruler and a pair of compasses they couldn't be measured exactly.

Surds are used to give exact answers instead of an approximation to many decimal places (in much the same way as fractions are preferred to decimals). Surds can also be manipulated using the following properties.

$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ Property 1 ◀ **Learn these properties.**

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ Property 2

$a\sqrt{b} + c\sqrt{b} = (a+c)\sqrt{b}$ Property 3

$a\sqrt{b} - c\sqrt{b} = (a-c)\sqrt{b}$ Property 4

Properties 3 and 4 demonstrate how surds behave in the process of factorisation.

Example 3

Simplify:

a $\sqrt{45}$ **b** $\sqrt{24}$ **c** $6\sqrt{7}+2\sqrt{7}$ **d** $5\sqrt{3}-\sqrt{27}$

Solution

a $\sqrt{45} = \sqrt{9 \times 5}$

$= \sqrt{9} \times \sqrt{5}$ ◀ **Using property 1.**

$= 3\sqrt{5}$

Note the square number, 9.

b $\sqrt{24} = \sqrt{4 \times 6}$

$= \sqrt{4} \times \sqrt{6}$ ◀ **Using property 1.**

$= 2\sqrt{6}$

Note the square number, 4.

c $6\sqrt{7}+2\sqrt{7} = (6+2)\sqrt{7}$ ◀ **Using property 3.**

$= 8\sqrt{7}$

d $5\sqrt{3}-\sqrt{27} = 5\sqrt{3}-\sqrt{9 \times 3}$

$= 5\sqrt{3} - (\sqrt{9} \times \sqrt{3})$ ◀ **Using property 1.**

$= 5\sqrt{3} - 3\sqrt{3} = 2\sqrt{3}$ ◀ **Using property 4.**

Notice how 27 can be written as a product involving a square number.

Example 4

Expand and simplify $(\sqrt{8}-\sqrt{3})(\sqrt{8}+\sqrt{3})$.

Solution

$$\begin{aligned}(\sqrt{8}-\sqrt{3})(\sqrt{8}+\sqrt{3}) &= \sqrt{8}(\sqrt{8}+\sqrt{3})-\sqrt{3}(\sqrt{8}+\sqrt{3})\\ &= (\sqrt{8})^2+\sqrt{8}\sqrt{3}-\sqrt{3}\sqrt{8}-(\sqrt{3})^2\\ &= (\sqrt{8})^2-(\sqrt{3})^2\\ &= 8-3=5\end{aligned}$$

Remember that $\sqrt{8}\sqrt{3}=\sqrt{3}\sqrt{8}$

Example 4 demonstrates the algebraic result for **difference of two squares:**

$(a+b)(a-b)=a^2-b^2$ ◀ **Learn this result.**

For example, $(\sqrt{8}+\sqrt{3})(\sqrt{8}-\sqrt{3})=(\sqrt{8})^2-(\sqrt{3})^2$, as demonstrated in Example 4. This technique will be very useful later.

Rationalising the denominator

Division by square roots can appear daunting, but it can be avoided by writing 1 in a surd form and multiplying by it. This allows the following technique to be applied.

Example 5

Rationalise:

a $\dfrac{5}{\sqrt{3}}$ **b** $\dfrac{15}{\sqrt{5}}$ **c** $\dfrac{1}{7-\sqrt{2}}$

Solution

a $$\begin{aligned}\frac{5}{\sqrt{3}} &= \frac{5}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{5\sqrt{3}}{3}\end{aligned}$$

Notice how 1 has been written in surd form as $\frac{\sqrt{3}}{\sqrt{3}}$.

b $$\begin{aligned}\frac{15}{\sqrt{5}} &= \frac{15}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}\\ &= \frac{15\sqrt{5}}{5}\\ &= 3\sqrt{5}\end{aligned}$$

Notice how 1 has been written in surd form.

Notice that 5 is a factor of both numerator and denominator.

This process is sometimes called **rationalising** the denominator because the original number is rewritten without the surd occurring in the denominator.

c The choice of surd form for 1 isn't obvious here. However the difference of two squares provides the answer.

$$\frac{1}{7-\sqrt{2}} = \frac{1}{(7-\sqrt{2})} \times \frac{(7+\sqrt{2})}{(7+\sqrt{2})}$$

Notice the form of 1.

$$= \frac{(7+\sqrt{2})}{(7)^2 - (\sqrt{2})^2}$$

The denominator is now the difference of two squares.

$$= \frac{7+\sqrt{2}}{49-2}$$

$$= \frac{7+\sqrt{2}}{47}$$

This choice of multiplier is by no means accidental. When more complicated expressions need rationalising the multiplier is simply the **conjugate** of the original denominator. (Think of the conjugate as having the same components, with one of the signs changed.)

Example 6

Rationalise $\dfrac{\sqrt{17}-\sqrt{5}}{\sqrt{17}+\sqrt{5}}$.

Solution

$$\frac{\sqrt{17}-\sqrt{5}}{\sqrt{17}+\sqrt{5}} = \frac{\sqrt{17}-\sqrt{5}}{\sqrt{17}+\sqrt{5}} \times \frac{(\sqrt{17}-\sqrt{5})}{(\sqrt{17}-\sqrt{5})}$$

Spot the conjugate.

$$= \frac{(\sqrt{17}-\sqrt{5})(\sqrt{17}-\sqrt{5})}{(\sqrt{17})^2 - (\sqrt{5})^2}$$

Spot the difference of two squares.

$$= \frac{(\sqrt{17})^2 - \sqrt{17}\sqrt{5} - \sqrt{5}\sqrt{17} + (\sqrt{5})^2)}{17-5}$$

Multiply out the numerator.

$$= \frac{17 + 5 - 2\sqrt{5}\sqrt{17}}{12}$$

Collect like terms.

$$= \frac{22 - 2\sqrt{85}}{12}$$

◀ **Use surd property 1.**

Factorise the numerator.

$$= \frac{2(11-\sqrt{85})}{12}$$

$$= \frac{11-\sqrt{85}}{6}$$

Richard Dedekind (1831–1916)
Dedekind showed that the real number line is continuous.

The real number system

The set of **real numbers** contains all the irrational numbers in addition to all the rational numbers. This set is denoted by $\mathbb{R}$. Real numbers can be represented by points on a line, called the real number line.

1.1 The Number System and Surds

Exercise

Technique

1 Simplify the following:

a $\sqrt{18}$
b $\sqrt{27}$
c $\sqrt{48}$
d $\sqrt{44}$
e $\sqrt{125}$
f $\sqrt{153}$
g $\sqrt{225}$
h $\sqrt{72}$

2 Simplify the following and express each as a single surd:

a $5\sqrt{3} - 2\sqrt{3}$
b $7\sqrt{2} + 2\sqrt{2}$
c $3\sqrt{18} - \sqrt{32}$
d $5\sqrt{32} - \sqrt{200}$
e $2\sqrt{27} + 3\sqrt{3} - \sqrt{12}$
f $2\sqrt{20} - \sqrt{45} + \sqrt{500}$

3 Expand and simplify the following:

a $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$
b $(\sqrt{3} - \sqrt{2})(\sqrt{3} - \sqrt{2})$
c $(\sqrt{5} - \sqrt{3})(\sqrt{5} - \sqrt{3})$
d $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$
e $(2\sqrt{5} + 1)(\sqrt{5} - 1)$
f $(4\sqrt{5} - 1)(2\sqrt{5} - 1)$
g $(\sqrt{7} - \sqrt{2})(\sqrt{7} + \sqrt{2})$
h $(\sqrt{11} - 3)(\sqrt{11} + 3)$

4 Rationalise the denominator in each of the following fractions:

a $\dfrac{1}{\sqrt{3}}$
b $\dfrac{3}{\sqrt{5}}$
c $\dfrac{1}{1 + \sqrt{5}}$
d $\dfrac{1}{\sqrt{2} - 1}$
e $\dfrac{7}{\sqrt{13} - \sqrt{11}}$
f $\dfrac{6}{2 + \sqrt{7}}$
g $\dfrac{\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}}$
h $\dfrac{\sqrt{11} - \sqrt{7}}{\sqrt{11} + \sqrt{7}}$

4 b

1.2 Indices

We can write $2 \times 2 \times 2 \times 2 \times 2$ as 2^5. The '5' is known as the **power** (sometimes called an exponent or index); 'a to the power x' is written as a^x. So a^x is an expression in which a is the **base** and x is the power.

Multiplication and division using powers

Consider multiplying the same number raised to different powers.

$$3^3 \times 3^2 = (3 \times 3 \times 3) \times (3 \times 3) = 3^5$$

$$4^2 \times 4^5 = (4 \times 4) \times (4 \times 4 \times 4 \times 4 \times 4) = 4^7$$

Try some of your own examples. What do you notice about the powers?

$a^p \times a^q = a^{(p+q)}$ Property 1   ◀ **Learn this property.**

Notice that when the base is the same, multiplication has the effect of summing the powers.
Now try multiplying a power by itself several times.

$$\begin{aligned}(3^3)^2 &= 3^3 \times 3^3 \\ &= (3 \times 3 \times 3) \times (3 \times 3 \times 3) \\ &= 3^6\end{aligned}$$

$$\begin{aligned}(4^2)^5 &= 4^2 \times 4^2 \times 4^2 \times 4^2 \times 4^2 \\ &= (4 \times 4) \times (4 \times 4) \times (4 \times 4) \times (4 \times 4) \times (4 \times 4) \\ &= 4^{10}\end{aligned}$$

Try some examples of your own. What is happening to the powers?

$(a^p)^q = a^{p \times q} = a^{pq}$ Property 2   ◀ **Learn this property.**

Notice that the powers are multiplying.
Now try division using powers. Remember to keep the base the same.

$$\begin{aligned}6^5 \div 6^3 &= \frac{6 \times 6 \times 6 \times 6 \times 6}{6 \times 6 \times 6} \\ &= 6 \times 6 = 6^2\end{aligned}$$

Try some examples of your own. What do you think is happening to the powers?

$a^p \div a^q = \dfrac{a^p}{a^q} = a^{p-q}$ Property 3   ◀ **Learn this property.**

Notice how the powers now subtract.

There are some special cases to consider.

1. Division when powers are equal

$$6^3 \div 6^3 = 6^{3-3}$$
$$1 = 6^0$$

Recall that any number divided by itself gives 1, except zero, which is a special case.

This result leads to another property.

Think of this as 'any number raised to the power zero must equal 1'.

$a^0 = 1$ **provided** $a \neq 0$ Property 4 ◀ **Learn this property.**

2. Division when the second power is larger than the first

$$7^3 \div 7^5 = 7^{3-5}$$
$$\frac{7 \times 7 \times 7}{7 \times 7 \times 7 \times 7 \times 7} = 7^{-2}$$
$$\frac{1}{7 \times 7} = 7^{-2}$$
$$\frac{1}{7^2} = 7^{-2}$$

Try some examples of your own. What does the negative power mean?

$a^{-p} = \dfrac{1}{a^p}$ Property 5 ◀ **Learn this property.**

A negative power indicates a reciprocal. (Think of a negative power as producing a fraction.)

3. Fractional powers

Property 1 can be used to introduce a meaning for a fractional power. Suppose that $p = q = \frac{1}{2}$ in property 1.

Notice that the left-hand side (LHS) is a number multiplied by itself.

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}}$$
$$(a^{\frac{1}{2}})^2 = a^1 = a$$
$$\Rightarrow \quad a^{\frac{1}{2}} = \sqrt{a}$$

Now take the square root of both sides.

The meaning of power $\frac{1}{3}$ can be established in a similar way.

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{(\frac{1}{3}+\frac{1}{3}+\frac{1}{3})}$$
$$(a^{\frac{1}{3}})^3 = a^1 = a$$
$$\Rightarrow \quad a^{\frac{1}{3}} = \sqrt[3]{a}$$

The LHS can be written more concisely using powers.

This means 'power $\frac{1}{3}$' can be thought of as a cube root. Investigate powers $\frac{1}{4}$, $\frac{1}{5}$, and so on in the same way. Notice how the fraction is related to the root.

$a^{\frac{1}{p}} = \sqrt[p]{a}$ Property 6 ◀ **Learn this property.**

Property 6 can also be used to establish a meaning for fractional powers where the numerator isn't 1. For example, $a^{2/3}$ can be written in other forms.

$$a^{\frac{2}{3}} = (a^2)^{\frac{1}{3}} = (a^{\frac{1}{3}})^2$$ ◀ Use property 2.

The first version, $a^{\frac{2}{3}} = (a^2)^{\frac{1}{3}}$, allows the following interpretation.

$$a^{\frac{2}{3}} = (a^2)^{\frac{1}{3}} = \sqrt[3]{a^2}$$

Notice how the fractional power creates a root of a power.

$$a^{\frac{p}{q}} = \sqrt[q]{a^p} = (a^{\frac{1}{q}})^p$$ Property 7 ◀ Learn this property.

These seven properties provide useful techniques for simplifying algebraic expressions.

Example 1

Simplify:

a $\dfrac{3^5 \times 3^6}{3^4}$ **b** $\dfrac{18x^2y^5}{3x^4y}$ **c** $(3x^5)^2$

Solution

a $\dfrac{3^5 \times 3^6}{3^4} = \dfrac{3^{5+6}}{3^4}$ ◀ Use property 1 on the numerator.

$= \dfrac{3^{11}}{3^4}$

$= 3^{11-4}$ ◀ Use property 3.

$= 3^7$

b $\dfrac{18x^2y^5}{3x^4y} = \dfrac{6x^2y^5}{x^4y}$

Divide the whole numbers.

$= 6x^{(2-4)}y^{(5-1)}$ ◀ Use property 3 on x and y separately.

$= 6x^{-2}y^4$

$= \dfrac{6y^4}{x^2}$ ◀ Use property 5.

c $(3x^5)^2 = 3^2 \times (x^5)^2$

Each term inside the bracket must be squared.

$= 9x^{10}$ ◀ Use property 2.

Example 2

Write the following expressions in index form:

a $\frac{2}{x^3}$ **b** $\frac{1}{2x^4}$ **c** $(x^4)^{\frac{1}{2}}$ **d** $\sqrt[3]{\frac{54x^4}{2x}}$

Solution

a $\frac{2}{x^3} = 2x^{-3}$ ◀ Use property 5.

b $\frac{1}{2x^4} = \frac{1}{2}x^{-4}$ ◀ Use property 5.

Notice in part **b** how the variable (x) is put into index form, but the fraction $(\frac{1}{2})$ is left as a multiplier.

c $(x^4)^{\frac{1}{2}} = x^{4\times\frac{1}{2}}$ ◀ Use property 2.

$= x^2$

d $\sqrt[3]{\frac{54x^4}{2x}} = \sqrt[3]{27x^{(4-1)}}$

Divide the number terms, and use property 3 on the variable x.

$= \sqrt[3]{27x^3}$

$= \sqrt[3]{27} \times \sqrt[3]{x^3}$ ◀ Use surd property 1.

$= 3x$

Notice that property 1 is true for cube roots too. This is because $(ab)^{\frac{1}{m}} = a^{\frac{1}{m}}b^{\frac{1}{m}}$

Example 3

Without using a calculator, evaluate:

a $9^{-\frac{3}{2}}$ **b** $(1\frac{11}{25})^{-\frac{1}{2}}$

Solution

a $9^{-\frac{3}{2}} = \frac{1}{9^{\frac{3}{2}}}$ ◀ Use property 5.

$= \frac{1}{(9^{\frac{1}{2}})^3}$ ◀ Use property 2.

Since $9^{\frac{1}{2}} = \sqrt{9} = 3$, by property 6.

$= \frac{1}{3^3} = \frac{1}{27}$

b $(1\frac{11}{25})^{-\frac{1}{2}} = (\frac{36}{25})^{-\frac{1}{2}}$

Remove the mixed number.

$= \frac{1}{(\frac{36}{25})^{\frac{1}{2}}}$ ◀ Use property 5.

$= \frac{1}{\sqrt{\frac{36}{25}}}$ ◀ Use property 6.

Remember that division by a fraction means invert the fraction (and multiply).

$= \frac{1}{(\frac{6}{5})} = \frac{5}{6}$

The properties, or rules, of indices can also be used in equation solving.

Example 4

Find the value of x when:

a $5^x = 125$ **b** $x^{\frac{2}{3}} = 4$

Solution

a $5^x = 125$

This type of problem can sometimes be solved using a trial and improvement approach, as follows, although this may be time consuming for very large numbers, and for non-integer solutions.

$5^1 = 5$

$5^2 = 5 \times 5 = 25$

$5^3 = 5 \times 5 \times 5 = 125$

So $x = 3$

b $x^{\frac{2}{3}} = 4$ ◀ **Use property 7.**

$\sqrt[3]{x^2} = 4$

$x^2 = 4^3$ Cubing both sides.

$x^2 = 64$

$x = \pm\sqrt{64}$ Taking the square root of both sides.

$x = \pm 8$

1.2 Indices

Exercise

Technique

1 Simplify the following expressions:

a $\dfrac{4^2 \times 4^7}{4^3}$

b $\dfrac{5^3 \times 5^4}{5^2}$

c $\dfrac{5^9}{5^2 \times 5^7}$

d $\dfrac{7^{11}}{7^3 \times 7^4}$

e $\dfrac{8x^3y^2}{4xy}$

f $\dfrac{12x^5y^4}{6x^3y^3}$

1 f

2 Simplify the following expressions:

a $(x^3)^4$

b $(x^2)^3$

c $2(x^3)^2$

d $3(x^5)^3$

e $(2x^4)^3$

f $(3x^2)^3$

3 Write the following expressions in index form:

a $\dfrac{3}{x^7}$

b $\dfrac{1}{3x^7}$

c $\dfrac{4^2 \times 4^7}{4^{12}}$

d $\dfrac{5^3 \times 5^5}{5^{10}}$

e $\dfrac{12x^2y^3}{24x^4y^7}$

f $\dfrac{3x^5y^3}{9x^2y^5}$

4 Write the following expressions in their simplest form:

a $\dfrac{(x^4)^{\frac{1}{2}}}{(x^6)^{\frac{1}{3}}}$

b $\dfrac{(x^8)^{\frac{1}{4}}}{x^{-2} \times x^4}$

c $\sqrt{\dfrac{9x^4}{y^2}}$

d $\sqrt{\dfrac{5x^6}{20x^2}}$

e $\sqrt[3]{\dfrac{16x^9}{2x^2 \times x}}$

f $\sqrt[4]{\dfrac{x^9 \times x^3}{(x^2)^2}}$

5 Evaluate the following expressions without using a calculator:

a $16^{\frac{3}{2}}$

b $64^{-\frac{2}{3}}$

c $(-27)^{-\frac{1}{3}}$

d $(-27)^{\frac{2}{3}}$

e $\left(\frac{9}{25}\right)^{-\frac{1}{2}}$

f $\left(\frac{9}{64}\right)^{-\frac{1}{2}}$

g $\left(3\frac{3}{8}\right)^{\frac{1}{3}}$

h $\left(2\frac{1}{4}\right)^{-\frac{3}{2}}$

5 e

6 Using trial and improvement, or otherwise, solve (find the value of x for) these equations:

a $2^x = 8$

b $3^x = 9$

c $7^x = \frac{1}{7}$

d $12^x = 1$

e $x^{-3} = \frac{1}{8}$

f $x^{-3} = \frac{27}{64}$

g $x^{-\frac{2}{3}} = \frac{1}{4}$

h $x^3 = 343$

1.3 Polynomials

An **expression** is a combination of numbers, variables (usually represented by the letters x and y) and mathematical operations (+, −, ×, ÷). Some examples of expressions are $3x + 2$, $7x^2 + 3y$, and $x^2 - 5x + 6$. When two expressions are linked by the symbol of equality an equation is formed.

Robert Recorde (1510–1558)
Recorde suggeste the equals sign, = in 1557.

For example, the following are all equations:

$$3x + 2 = 5 \qquad 7x^2 + 3y = x + 2 \qquad x^2 - 5x + 6 = 0$$

The general form of a simple equation can be written in various ways.

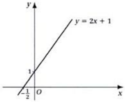
$$y = ax + b, \quad y = mx + c \quad \text{and} \quad y = a_1x + a_0$$

are all forms of **linear equations.**

a, b, m, c, a_1 and a_0 ar constants distinct fro x and y, which represent variables.

Why are these equations linear? Try drawing their graphs for some values of a and b, m and c, or a_1 and a_0. Notice how the graph is always a straight line.

For example, $y = 2x + 1$ is one version of $y = ax + b$, with $a = 2$ and $b = 1$. It is illustrated here.



An equation of the form $y = x^n$, where n is a positive integer, is known as a **polynomial**. The general form of this type is written $y = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where $a_n \neq 0$. This is called a polynomial equation of degree n.

The word polynomial Greek, from 'poly' meaning many and 'nomial' meaning tern or names.

$$y = x^2 + 3x - 1$$

is a polynomial of degree 2, because its highest power is 2.

$x^2 + \sqrt{x} - 3$ is not a polynomial. This is because $\sqrt{x} = x^{\frac{1}{2}}$ and isn't a positive integer

$$y = 2x^5 + 7x^3 - 8x + 4$$

is a polynomial of degree 5, because its highest power is 5.

What about the numbers in the polynomials? Notice how each term of the equation is distinct, separated by the operators + and −. The number in front of each **variable** (letter) is called a **coefficient**.

Consider the polynomial $y = 2x^5 + 7x^3 - 8x + 4$. This is a quintic equation (order 5, because the highest power of x is 5). The numbers 2, 7 and -8 are coefficients: 2 is the coefficient of the x^5 term; 7 is the coefficient of the x^3 term; and -8 is the coefficient of the x term. The final term, $+4$, is called a **constant** because it remains the same (its value is constant), whatever the value of x.

Addition and subtraction

In order to simplify polynomials, like terms are collected together. Like terms can then be added or subtracted algebraically. Notice the distinction between like and unlike terms.

Remember like terms contain the same variables (letters) raised to the same powers.

- *Examples of like terms*

  x^2, $-3x^2$, and $2x^2$ are like terms, because they all have the same variable, x^2

  xy and $3yx$ are like terms, because they both have the same variable, xy

- *Examples of unlike terms*

  x and xy are unlike terms, because they contain different variables

  x^2y and y^2x are unlike terms, because they contain different variables

Example 1

Simplify:

a $(3x^2 + 8x - 2) + (5x^2 + 3x + 8)$

b $(6x^2 + 8x - 2) - (5x^2 + 2x - 7)$

Solution

a $(3x^2 + 8x - 2) + (5x^2 + 3x + 8) = 3x^2 + 8x - 2 + 5x^2 + 3x + 8$

$= 3x^2 + 5x^2 + 8x + 3x - 2 + 8$

Collect together the like terms.

$= 8x^2 + 11x + 6$

Combine the like terms.

b $(6x^2 + 8x - 2) - (5x^2 + 2x - 7) = 6x^2 + 8x - 2 - 5x^2 - 2x + 7$

$= 6x^2 - 5x^2 + 8x - 2x - 2 + 7$

$= x^2 + 6x + 5$

Recall that a minus sign outside the bracket changes the sign of each term inside the bracket, and collect together like terms.

Multiplication

Multiplication of polynomials is achieved by applying the **distributive law**. In algebra this law can be stated,

$$\boldsymbol{a(b + c) = ab + ac}$$

and

$$\boldsymbol{(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd}$$

Verify this law for yourself using numbers.

For example, 23×17 can be written

$$(20+3) \times (10+7) = 20(10+7) + 3(10+7)$$
$$= 200 + 140 + 30 + 21 = 391$$

What is the effect of this law? Notice that every term in the second bracket has been multiplied by each of the terms in the first bracket. This is the principle used in polynomial multiplication.

Example 2

Expand and simplify:

a $(2x+3)(4x^3+3x^2-2x+5)$
b $(3x^3-x^2+7)(2x^3+2x^2-3x+2)$ **c** $(x+2)^3$

Solution

a Recall the structure of the distributive law. Each term in the second bracket must be multiplied by each term in the first bracket.

$$(2x+3)(4x^3+3x^2-2x+5)$$
$$= 2x(4x^3+3x^2-2x+5) + 3(4x^3+3x^2-2x+5)$$
$$= 8x^4+6x^3-4x^2+10x+12x^3+9x^2-6x+15$$
$$= 8x^4+6x^3+12x^3-4x^2+9x^2+10x-6x+15$$
$$= 8x^4+18x^3+5x^2+4x+15$$

Multiply the second bracket by $2x$ and by 3

Collect like terms.

b Recall the structure of the distributive law.

$$(3x^3-x^2+7)(2x^3+2x^2-3x+2)$$
$$= 3x^3(2x^3+2x^2-3x+2) - x^2(2x^3+2x^2-3x+2)$$
$$+ 7(2x^3+2x^2-3x+2)$$
$$= 6x^6+6x^5-9x^4+6x^3-2x^5-2x^4+3x^3-2x^2$$
$$+ 14x^3+14x^2-21x+14$$
$$= 6x^6+6x^5-2x^5-9x^4-2x^4+6x^3+3x^3+14x^3-2x^2+14x^2$$
$$-21x+14$$
$$= 6x^6+4x^5-11x^4+23x^3+12x^2-21x+14$$

Notice how the indice combine.

Collect like terms.

c $(x+2)^3 = (x+2)(x+2)(x+2)$ ◀ **Using the distributive law on the second pair of brackets.**

$$= (x+2)[x(x+2)+2(x+2)]$$
$$= (x+2)[x^2+2x+2x+4]$$
$$= (x+2)(x^2+4x+4)$$
$$= x(x^2+4x+4) + 2(x^2+4x+4)$$
$$= x^3+\cancel{4}x^2+4x+\cancel{2}x^2+8x+8$$
$$= x^3+6x^2+12x+8$$

Simplify the expressi in square brackets.

Notice how the faint crossing out can help collect and combine li terms quickly.

Division

Before attempting division of polynomials think back to division with integers (whole numbers). Numerical division can be represented by fractions.

Fractions in the form $\frac{3}{4}, \frac{1}{10}, \frac{2}{5}, \frac{3}{8}$ are known as **proper fractions**. Here the numerator (top number) is less than the denominator (bottom number).

How are **improper fractions** written? Here the reverse is true. The numerator is greater than the denominator. Some examples are $\frac{7}{4}, \frac{11}{10}, \frac{18}{5}$, and $\frac{19}{8}$. How else could these be written? Improper fractions can also be written as mixed numbers, that is a mixture of an integer (whole number) and a proper fraction.

$$\tfrac{7}{4} = 1\tfrac{3}{4}, \quad \tfrac{11}{10} = 1\tfrac{1}{10}, \quad \tfrac{18}{5} = 3\tfrac{3}{5} \quad \text{and} \quad \tfrac{19}{8} = 2\tfrac{3}{8}.$$

Similarly **algebraic fractions** (fractions involving polynomials) can be both proper and improper.

$$\frac{3}{x+1}, \ \frac{2}{x-5}, \ \frac{x+1}{x^2+5x+3} \quad \text{are all proper algebraic fractions.}$$

$$\frac{x+2}{x+5}, \ \frac{x^2+13}{x-4}, \ \frac{x^2+5x+6}{x^2-7x+12} \quad \text{are all improper algebraic fractions.}$$

What's the difference between proper and improper algebraic fractions? Look at the degree, or order, of the polynomials in the numerator and denominator. For an algebraic fraction to be proper the order of the numerator must be *less than* the order of the denominator. If not, the fraction is improper.

Example 3

Make the expression $\dfrac{x+3}{x+5}$ a proper algebraic fraction.

Solution

Notice that $\dfrac{x+3}{x+5}$ is improper (the numerator and denominator are both order 1, or linear). There are two techniques that could be used to make this a proper algebraic fraction.

Here the 'x's cannot be cancelled because they are not factors of both numerator and denominator.

Method 1: The division algorithm

Write $\dfrac{x+3}{x+5}$ as a division problem, and try a division by x.

$$x+5\overset{\displaystyle 1}{\overline{\smash{)}x+3}}$$

$$\begin{array}{r} 1 \\ x+5\overline{\smash{)}x+3} \\ \underline{x+5} \\ -2 \end{array}$$

So $\dfrac{x+3}{x+5} = 1 - \dfrac{2}{x+5}$

Because x in $(x+5)$ w… divide into the x in $(x+3)$ once.
Subtract $1 \times (x+5)$ from $(x+3)$. The remainder is $3 - 5 = -2$.

Notice the positions of the remainder and the divisor in the result.

Method 2: Algebraic manipulation
In this technique the numerator is written as a multiple of the denominator and a remainder.

$$\frac{x+3}{x+5} = \frac{(x+5)-2}{x+5}$$

Then the numerator is split into two distinct parts.

$$\frac{x+3}{x+5} = \frac{x+5}{x+5} - \frac{2}{x+5} = 1 - \frac{2}{x+5}$$

Because a number, $(x+5)$, divided by its… is always 1.

Notice how both techniques give the same result. The fraction in the answer is proper.

Example 4

Write as proper fractions:

a $\dfrac{4x}{1-x}$ **b** $\dfrac{3x+6}{x+3}$

Solution

a Using the division algorithm,

$$\begin{array}{r} -4 \\ 1-x\overline{\smash{)}4x} \\ \underline{4x-4} \\ 4 \end{array}$$

$$\frac{4x}{1-x} = -4 + \frac{4}{(1-x)}$$

Because $4x \div -x = -$…
Then $-4 \times (1-x) =$ … $-4 + 4x = 4x - 4$.
Finally, $0 - (-4) = 4$.

b $\dfrac{3x+6}{x+3} = \dfrac{3(x+3)-3}{x+3}$

$$= \frac{3(x+3)}{x+3} - \frac{3}{x+3}$$

$$= 3 - \frac{3}{x+3}$$

The coefficient of x in the numerator is 3, so $(x+3)$ is multiplied b… 3. Multiplying out the bracket, the constant term in the numerator… 9. We require it to be $+6$, so we subtract 3.

1.3 Polynomials
Exercise
Technique

1 For each of the following polynomials write down (**i**) the order (degree) and (**ii**) the coefficient of x^2:

a $3x^3 + 2x^2 - 1$
b $5x^4 - 2x^2 + x$
c $7x^5 - 3x^4 - 7x^2 + 9$
d $3x^2 - 7$
e $3x^3 - 6x + 9$
f $8x^2 - 2x + 3$

2 Add:

a $3x^2 + 7x - 3$ and $5x^2 - 2x + 8$
b $6x^2 + 6x + 3$ and $6x^2 + 3x - 2$
c $4x^3 + 2x^2 + 3x + 6$ and $3x^3 + 7x - 3$
d $12x^5 + 7x^3 - 3x + 9$ and $-3x^5 + 2x^4 + 7x^2 + 7x$

3 Find $y_1 - y_2$ when:

a $y_1 = 5x^2 + 12x + 3$ and $y_2 = 3x^2 + 7x - 4$
b $y_1 = 7x^2 + 12x - 2$ and $y_2 = 2x^2 - 5x + 7$
c $y_1 = 5x^4 + 3x^3 - 2x^2 + 8x$ and $y_2 = 7x^4 - 2x^3 - 2x^2 + 2x - 5$
d $y_1 = -3x^3 - 2x^2 + 7x + 13$ and $y_2 = 5x^3 - 3x^2 + 6x + 12$

4 Multiply and simplify the following:

a $(2x + 1)$ and $(3x + 2)$
b $(2x + 3)$ and $(2x^2 + 1)$
c $(2x - 1)$ and $(x^2 + 3x + 1)$
d $(3x - 1)$ and $(x^2 - 2x + 1)$
e $(3 - x)$ and $(x^2 + 3x - 2)$
f $(4 - x)$ and $(2x^2 - 5x + 7)$

4 a

5 Expand and simplify the following:

a $(2x + 3)(3x^2 - 2x + 8) + (x + 1)(x^2 + 3x + 2)$
b $(5x - 1)(2x^2 + 3x + 2) + (x + 3)(2x^2 + 4x - 3)$
c $(3x - 2)(3 + 2x - x^2) + (x - 5)(x^2 - 2x + 1)$
d $(4x - 3)(2x + 7 - 2x^2) + (x - 1)(x^2 + 5x - 1)$
e $(5x + 1)(x^2 + 2x + 2) - (x + 1)(x^2 + 3x + 1)$
f $(3x - 2)(2x^3 + 7x - 5) - (x - 1)(x^3 - 3x + 2)$

6 Expand and simplify the following:

a $(2x + 3)(2x + 3)$
b $(2x + 3)(2x + 3)^2$
c $(2x + 3)^4$
d $(1 + 2x)^4$
e $(x + 1)^4$
f $(x - 1)^4$

7 Write the following as proper fractions:

a $\dfrac{x + 2}{x + 5}$
b $\dfrac{2x + 2}{x + 5}$
c $\dfrac{4x + 10}{x - 1}$
d $\dfrac{6x + 7}{2x - 1}$
e $\dfrac{3x - 2}{x + 2}$
f $\dfrac{2x + 3}{x - 2}$

1.4 Factorisation

The distributive law $a(b + c) = ab + ac$ has been used to demonstrate multiplication of polynomials. The same law can be used in reverse, so a sum of terms can be written as a product. Doing this often introduces brackets into the algebra. The process is called **factorisation.**

Factorisation can be shown with natural numbers. The integers that divide exactly into 8 are 1, 2, 4 and 8. These are called the **factors** of 8. Notice that 8 can be written as a product of some of these factors.

$$8 = 1 \times 8 \qquad 8 = 2 \times 4$$

Polynomials too can sometimes be factorised. Consider first polynomials of degree 2. These are more commonly known as **quadratics**. One such quadratic expression is $x^2 + 3x - 18$. It can be factorised as follows.

$$x^2 + 3x - 18 = (x + 6)(x - 3)$$

Check this by multiplying out the brackets. We call $(x + 6)$ and $(x - 3)$ the **factors** of the quadratic $x^2 + 3x - 18$.

Quadratics can be factorised using one of three basic techniques:

- extracting a common factor
- trial and improvement
- standard results – difference of two squares.

Extracting a common factor

Example 1

Factorise $4x^3y^2 - 8x^2y^3$.

Factorisation by extracting a common factor is not restricted to quadratics.

Solution

Here it is possible to extract common factors.
4 and 8 have the common factor 4 (4 is the largest factor of both numbers).
x^3 and x^2 have the common factor x^2 (x^2 is the largest factor of both terms).
y^2 and y^3 have the common factor y^2 (y^2 is the largest factor of both terms).
So $4x^3y^2 - 8x^2y^3 = 4x^2y^2 \times$ (some other term).
Notice how the common factors are extracted from both terms in the expression and appear outside the bracket. These form one part of the product. The bracket must contain the terms necessary to combine with the common factor to create the original expression.

$$4x^3y^2 - 8x^2y^3 = 4x^2y^2(x - 2y)$$

Check that $4x^2y^2 \times x = 4x^3y^2$ and that $4x^2y^2 \times (-2y) = -8x^2y^3$.

In order to factorise quadratics, first check the coefficients in the expression.

Consider again the quadratic expression, $x^2 + 3x - 18$. The coefficient of x^2 is 1, the coefficient of x is 3 and the constant term is -18. So the three distinct numbers in this expression are 1, 3 and -18. Now look at the factors of that expression, $(x + 6)$ and $(x - 3)$. These contain the two distinct numbers 6 and -3. How are these two sets of numbers connected? Notice that adding the numbers in the factors and multiplying the numbers in the factors creates the two larger numbers given by the coefficients.

$6 + (-3) = 3$, which is the coefficient of x

$6 \times (-3) = -18$, which is the constant term.

The general rule, for a quadratic expression where the coefficient of x^2 is 1, is that the expression can be factorised if two numbers can be found that add to give the coefficient of x and multiply to give the constant term.

Trial and improvement

Example 2

Factorise:

a $x^2 + 7x + 12$ **b** $x^2 - 8x + 15$

Solution

a The coefficient of x^2 is 1, so we know that we need to find two numbers that add to make 7 and multiply to make 12. Notice that 4 and 3 work.

$x^2 + 7x + 12 = (x + 4)(x + 3)$

Check this by multiplying out the factors.

b The coefficient of x^2 is 1, so we need to find two numbers that add to -8 and multiply to 15. We find that -5 and -3 work

$-5 + (-3) = -8$

$(-5) \times (-3) = 15$

So $x^2 - 8x + 15 = (x - 5)(x - 3)$.

Check this by multiplying out the factors.

By finding the factors of the constant term first, much of the trial and improvement in these examples can be done quickly.

What happens when the coefficient of x^2 is greater than 1? This trial and improvement technique can then be modified. The method, or process, is sometimes known as 'PAFF', the letters P, A, F and F representing the four stages of the process: Product, Addition, Factors and Factorise. The idea is to change the algebra into smaller numerical problems leading to some less complicated factorisation.

Follow each stage of the technique in Example 3.

Example 3

Factorise $12x^2 + 17x - 14$.

Solution

Notice that the coefficient of x^2 is 12 so the technique used in Example 2 won't work. Try PAFF, the stages of which are as follows.

1. P – Product

   Multiply the coefficient of x^2 by the constant term.

   Here $P = 12 \times (-14)$

   $P = -168$

2. A – Addition

   This is the coefficient of x, something that factors need to add to.

   Here, $A = 17$

3. F – Factors

   Using the same technique as before, but this time find two numbers that multiply to give P and add to give A.

   In this example -7 and 24 work.

   $(-7) \times 24 = -168$

   $-7 + 24 = 17$

4. F – Factorise

   Use the factors identified in Step 3 to help factorise the expression. These allow the coefficient of x to be split.

   $12x^2 + 17x - 14 = 12x^2 - 7x + 24x - 14$

   Notice how the factors in Step 3 are used to produce a four-term expression from the original three-term quadratic.

   This new expression can now be factorised by extracting common factors. Imagine factorising the first pair of terms and the second pair of terms separately.

   $12x^2 - 7x + 24x - 14 = x(12x - 7) + 2(12x - 7)$

   Notice that $(12x - 7)$ is a new common factor.

   $12x^2 - 7x + 24x - 14 = (12x - 7)(x + 2)$

   So $12x^2 + 17x - 14 = (12x - 7)(x + 2)$

   Check by multiplying out the factors.

This process looks complicated and time consuming, but with practice can be a very effective algorithm for factorising quadratics where the coefficient of x^2 is greater than 1.

Example 4

Factorise $4x^2 - 2x - 30$.

Solution

The coefficient of x^2 is 4, so use PAFF.

An alternative method here would be to extract the common factor first and use PAFF on a simpler equation.

P: $4 \times (-30) = -120$

A: the coefficient of x is -2

F: the factors need to multiply to -120, and add to -2. Check that -12 and 10 work.

F:
$$\begin{aligned} 4x^2 - 2x - 30 &= 4x^2 - 12x + 10x - 30 \\ &= 4x(x-3) + 10(x-3) \\ &= (x-3)(4x+10) \end{aligned}$$

Factorise the first and second pair of terms separately.

Notice that $(x - 3)$ is a new common factor.

So $4x^2 - 2x - 30 = (x-3)(4x+10)$

What do you notice about the second factor? The numbers 4 and 10 have 2 as a common factor, so this bracket can be factorised further.

$$4x + 10 = 2(x+5)$$

So the original quadratic expression has three distinct factors.

$$4x^2 - 2x - 30 = 2(x+5)(x-3)$$

Check this by multiplying out the factors.

Difference of two squares

Sometimes the coefficient of x can be zero. In this case the quadratic will contain an x^2 term and a constant term only. Factorisation of these expressions can often be achieved by extracting a common factor or using another standard result: the **difference of two squares**.

Example 5

Factorise $x^2 - 49$.

Solution

Notice that 49 is a square number; that is, $49 = 7^2$. The expression can therefore be rewritten,

$$x^2 - 49 = x^2 - 7^2$$

The right-hand side is now the difference of two squares. This factorises in a particular way.

$$\begin{aligned} x^2 - 49 &= x^2 - 7^2 \\ &= (x+7)(x-7) \end{aligned}$$

The result demonstrated in Example 5 can be generalised as

$$a^2 - b^2 = (a + b)(a - b)$$

This result can be used as an aid to computation. Some 'difficult' problems can be done quickly without using a calculator.

Example 6

a Factorise $12x^2 - 3$.

b Factorise $5\tan^2\theta - 5$.

c Evaluate $101^2 - 100^2$.

Solution

a $12x^2 - 3 = 3(4x^2 - 1)$

$= 3[(2x)^2 - 1^2]$

$= 3(2x + 1)(2x - 1)$

Extract the common factor 3. The bracketed factor is the difference of two squares.

b $5\tan^2\theta - 5 = 5(\tan^2\theta - 1)$

$= 5(\tan^2\theta - 1^2)$

$= 5(\tan\theta + 1)(\tan\theta - 1)$

c $101^2 - 100^2 = (101 + 100)(101 - 100)$

$= 201 \times 1$

$= 201$

Use your calculator to check that $101^2 - 100^2 = 201$.

1.4 Factorisation

Exercise

Technique

1 Factorise the following quadratic expressions:

a $x^2 + 3x + 2$
b $x^2 + 7x + 10$
c $x^2 - x - 20$
d $x^2 - 7x - 18$
e $x^2 + 3x - 18$
f $x^2 + x - 12$
g $x^2 + 6x - 16$
h $x^2 + x - 6$

1 g

2 Factorise the following quadratic expressions using the difference of two squares:

a $x^2 - 16$
b $y^2 - 9$
c $9x^2 - 1$
d $16y^2 - 1$
e $\cos^2\theta - 1$
f $\sin^2\theta - 1$
g $4x^2 - 25y^2$
h $81x^2 - 36y^2$

3 Factorise the following expressions completely:

Hint: Remove the common factor first.

a $2x^2 - 32$
b $3y^2 - 27$
c $20x^2 - 5$
d $50y^2 - 200$
e $2t^3 - 450t$
f $2\cos^2\theta - 2$

4 Factorise the following expressions:

a $3x^2 + x - 2$
b $2x^2 - 5x + 3$
c $7x^2 + 22x + 3$
d $4x^2 - 12x + 5$
e $10x^2 - 41x - 45$
f $8x^2 - 21x - 9$
g $8x^2 - 17x + 9$
h $6x^2 - 7x - 3$

4 b

5 Factorise the following expressions:

a $4x^2 - 10x + 6$
b $x^2 + 2xy - 8y^2$
c $x^2 + 5xy - 36y^2$
d $10x^2 - 18x - 4$
e $36x^2 - 33x + 6$
f $16x^2 - 100x + 150$
g $2x^2y - 7xy + 3y$
h $60x^2y - 55xy - 25y$

1.5 Solving Quadratic Equations

As in a quadratic expression, a quadratic equation in one unknown has the variable occurring at least once raised to the second power. The variable doesn't occur to any higher powers. Some examples of quadratic equations are $x^2 - 4 = 0$, $a^2 + 4a - 5 = 0$, $2t^2 - 16t + 36 = 0$, $p^2 + p = 2$.

Greek mathematicians solved algebraic problems like these using geometry. They could solve all quadratic equations that had real number solutions. See Book X of the 'Elements' of Euclid (450–380 BC).

Quadratic equations often occur in the solution of real-life problems, such as in echo sounding, calculating depths of wells and hardness testing.

Can all quadratic equations be solved? The main techniques used to solve quadratic equations are:

- factorising
- completing the square
- using the quadratic formula
- graphical methods.

Quadratic equations generally have two solutions, which can be distinct or repeated.

Factorising quadratic equations

Example 1

Solve $(x - 5)(x + 2) = 0$.

Solution

Notice that the left-hand side of this equation is a product of two factors. The result of multiplying these factors is zero. *If two quantities multiply to zero then one of them must be zero.*
Since $(x - 5)(x + 2) = 0$, then $(x - 5) = 0$ or $(x + 2) = 0$.
These linear equations can now be solved.

$$x - 5 = 0 \Rightarrow x = 5$$
$$\text{and } x + 2 = 0 \Rightarrow x = -2$$

The symbol $\Rightarrow$ means 'implies'.

Check that these are solutions by substituting them back into the original equation. Notice that both of these values of x are solutions; there are two solutions.

Example 2

Solve:

a $x^2 - 4x - 5 = 0$
b $2x^2 - 32 = 0$
c $x^2 - 3x = 0$

Solution

a First factorise the quadratic expression.

$$x^2 - 4x - 5 = 0$$

$$(x + 1)(x - 5) = 0$$

So $(x + 1) = 0$ or $(x - 5) = 0$

$$x = -1 \text{ or } x = 5$$

The coefficient of x^2 is 1.
$(+1) + (-5) = -4$,
$(+1) \times (-5) = -5$

Substitute both values of x separately into the original equation; $x = -1$ and $x = 5$ are its solutions.

b $2x^2 - 32 = 0$

$$2x^2 - 32 = 2(x^2 - 16)$$

$$= 2(x^2 - 4^2) = 2(x + 4)(x - 4)$$

So $2x^2 - 32 = 2(x + 4)(x - 4) = 0$

Recognise that $16 = 4^2$, and then use the difference of two squares.

Now there are three factors multiplying to give a zero result.

Since $2 \neq 0$, $(x + 4) = 0$ or $(x - 4) = 0$

So $x = -4$ or $x = 4$

Both solutions satisfy $2x^2 - 32 = 0$.

c $x^2 - 3x = x(x - 3) = 0$

So $x = 0$ or $(x - 3) = 0$

$x = 0$ or $x = 3$

Both solutions satisfy $x^2 - 3x = 0$.

Example 3

Solve the quadratic equation $12x^2 + 17x - 14 = 0$.

Solution

$$12x^2 + 17x - 14 = 0$$

In this example the coefficient of x^2 isn't 1, so try PAFF.

This particular quadratic expression was factorised in Example 3 of Section 1.4.

$$12x^2 + 17x - 14 = (12x - 7)(x + 2) = 0$$

So $(12x - 7) = 0$ or $(x + 2) = 0$

$x = \frac{7}{12}$ or $x = -2$

Notice that one of the solutions, $\frac{7}{12}$, isn't an integer (whole number). Its fractional value is exact, but the decimal representation is recurring. Check this using a calculator.

Completing the square

Sometimes the quadratic expression cannot be factorised easily. The equation may then be solved using a technique called **completing the square**.

To complete the square we need to write the quadratic in the form $(x + a)^2 = b$, where a and b are real numbers. Then the value of x can be found by taking the square root of both sides of the equation. Writing the quadratic in this form requires some skill in algebraic manipulation.

Example 4

Solve the equation $x^2 - 6x - 5 = 0$.

Solution

Try factorisation. What happens? Notice that no pair of integers add to give -6 and multiply to give -5. Don't despair: complete the square. First rewrite the equation, separating the variables from the constant term.

$$x^2 - 6x - 5 = 0$$

So $$x^2 - 6x = 5$$

$$x^2 - 6x + 9 = 5 + 9 = 14$$

So $$(x-3)(x-3) = 14$$

That is, $$(x-3)^2 = 14$$

$$x - 3 = \pm\sqrt{14}$$

We need the LHS to be a perfect square (we want to write it in the form $(x+a)^2$), so add 9 to both sides.
We can now factorise the LHS because $(-3) + (-3) = -6$ and $(-3) \times (-3) = 9$.

Recall that any positive square number has a positive square root and a negative square root.

The solutions are therefore $x = 3 \pm \sqrt{14}$.

Notice that the steps in the process of completing the square are:

Step ① Separate the constant term from the variable terms.
Step ② Add a value to each side of the equation to force one side to be a perfect square.

How did you know what value to add? There is a simple rule. Provided the coefficient of x^2 is 1, simply halve the coefficient of x and then square this value.

Example 5

Solve $x^2 - 3x - 5 = 0$.

Solution

$x^2 - 3x - 5 = 0 \Rightarrow x^2 - 3x = 5$ ◀ **① Separate the terms.**

First check that factorisation doesn't work easily.

The coefficient of x is -3. Half this is $-\frac{3}{2}$. Squaring that, we have $\frac{9}{4}$. So add $\frac{9}{4}$ to both sides of the equation.

$x^2 - 3x + \frac{9}{4} = 5 + \frac{9}{4}$ ◀ **② Force one side to be a perfect square.**

The left-hand side is now a perfect square, so we can factorise it.

$(x - \frac{3}{2})^2 = 5 + \frac{9}{4} = \frac{29}{4}$

Take the square root of both sides.

$$x - \frac{3}{2} = \pm\sqrt{\frac{29}{4}}$$

$$x = \frac{3}{2} \pm \frac{\sqrt{29}}{2} = \frac{3 \pm \sqrt{29}}{2}$$

Recall the properties of surds.

Notice that written in this form (a surd), we have exact solutions and not decimal approximations. The solutions have also been written concisely with a common denominator.

Remember that this technique only works when the coefficient of x^2 is 1. When it isn't, divide each term in the equation by the coefficient of x^2. This often creates equations with fractions as coefficients.

Example 6

Solve $3x^2 + 4x - 5 = 0$.

Solution

The coefficient of x^2 is 3, so divide each term by 3.

$$3x^2 + 4x - 5 = 0 \Rightarrow x^2 + \tfrac{4}{3}x - \tfrac{5}{3} = 0$$

Now the algorithm can be used as before.

$x^2 + \tfrac{4}{3}x = \tfrac{5}{3}$ ◀ ① **Separate the terms.**

Add $\tfrac{4}{9}$ to both sides of the equation (recall that this number is reached by halving the coefficient of x, and then squaring the result).

$x^2 + \tfrac{4}{3}x + \tfrac{4}{9} = \tfrac{5}{3} + \tfrac{4}{9}$ ◀ ② **Force one side to be a perfect square.**

$$(x + \tfrac{2}{3})^2 = \tfrac{19}{9}$$

$$x + \tfrac{2}{3} = \pm\sqrt{\frac{19}{9}} = \pm\frac{\sqrt{19}}{3}$$

$$x = \frac{-2 \pm \sqrt{19}}{3}$$

Check with a calculator that these values satisfy the original equation.

Sometimes the quadratic equation doesn't have to be solved. It may be sufficient to write it in the form of a perfect square. Suppose that $y = ax^2 + bx + c$ can be written in the form $y = a(x+p)^2 + q$, where a, b, c, p and q are all real numbers.

$$\begin{aligned} \text{Then } ax^2 + bx + c &= a(x+p)^2 + q \\ &= a(x^2 + 2px + p^2) + q \\ ax^2 + bx + c &= ax^2 + 2apx + ap^2 + q \end{aligned}$$

By comparing coefficients you should see a relationship between a, b, c and p, q. Compare the coefficients of x; that is, see how many 'x's there are on each side of the equation.

$$b = 2ap$$

Comparing the constant terms in the same way,

$$c = ap^2 + q$$

Since a, b and c are already known (directly from the quadratic) these two results can sometimes be used to establish p and q quickly.

Example 7

Express the following in the form $a(x + p)^2 + q$:

a $x^2 - 3x - 5$ **b** $-5x^2 - 2x + 3$

Solution

a Notice that $a = 1$, $b = -3$ and $c = -5$.
We require $x^2 - 3x - 5 = a(x + p)^2 + q$.
Since $a = 1$, this can be simplified to

$$\begin{aligned} x^2 - 3x - 5 &\equiv (x + p)^2 + q \\ &\equiv (x^2 + 2px + p^2) + q \\ &\equiv x^2 + 2px + p^2 + q \end{aligned}$$

The symbol $\equiv$ is used show that these expressions are equivalent for all valu of x.

Now comparing coefficients of x and constant terms,

$$-3 = 2p \quad \text{and} \quad -5 = p^2 + q$$

Then $p = -\frac{3}{2}$

and $q = -5 - p^2 = -5 - \frac{9}{4} = -\frac{29}{4}$

So $x^2 - 3x - 5 \equiv (x - \frac{3}{2})^2 - \frac{29}{4}$

Notice that when $x =$ $(x - \frac{3}{2})^2 = 0$. This mea the quadratic expressi has a least value of $-$

b Let

$$\begin{aligned} -5x^2 - 2x + 3 &\equiv a(x + p)^2 + q \\ &\equiv a(x^2 + 2px + p^2) + q \\ &\equiv ax^2 + 2apx + ap^2 + q \end{aligned}$$

Comparing coefficients of x^2, x and the constant terms,

$$-5 = a, -2 = 2ap \quad \text{and} \quad 3 = ap^2 + q$$

Since $a = -5$, $-2 = 2ap$ becomes $-2 = 2 \times (-5) \times p = -10p$
So $p = \frac{1}{5}$

$$-5x^2 - 2x + 3 \equiv -5(x + \tfrac{1}{5})^2 + q$$

Now $3 = ap^2 + q$

Check this by multiplying out the bracket and collecting like terms.

So $3 = -5(\frac{1}{5})^2 + q$

$$q = 3 + \tfrac{5}{25} = 3 + \tfrac{1}{5} = \tfrac{16}{5}$$

So $-5x^2 - 2x + 3 \equiv -5(x + \frac{1}{5})^2 + \frac{16}{5}$

Verify that $\frac{16}{5}$ is the maximum value of this expression.

The quadratic formula

By completing the square on the general quadratic expression $ax^2 + bx + c$, we can create a formula that can be used to solve quadratic equations simply by substituting values for a, b and c.
Suppose $ax^2 + bx + c \equiv 0$ for some real values of a, b and c.

Then $x^2 + \dfrac{bx}{a} + \dfrac{c}{a} = 0.$

Divide by a to make the coefficient of x^2 equal to 1.

Now complete the square on this expression in the same way as before.

$$x^2 + \frac{bx}{a} = -\frac{c}{a}$$ ◀ ① Separate the terms.

The value to add to both sides of the equation is found, as before, by halving the coefficient of the x term, and then squaring the result.

$$x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$ ◀ ② Force one side to be a perfect square.

Recall that the LHS is now a perfect square.

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ ◀ Learn this result.

Graphical calculator support pack

Once a, b and c have been identified, this formula can be used to solve quadratic equations.

Example 8

Solve $x - 7 = \frac{4}{x}$.

Solution

This may not at first appear to be a quadratic equation, but multiplying both sides by x gives $x(x - 7) = 4$. Notice how the denominator has multiplied the whole of the left-hand side of the equation. Now multiply $x(x - 7)$ out to give $x^2 - 7x = 4$. Moving the constant term, the original equation has been rearranged to the form for which the formula works.

$$x^2 - 7x - 4 = 0$$

Now, $a = 1$, $b = -7$ and $c = -4$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Always quote the formula.

$$x = \frac{7 \pm \sqrt{49 - 4 \times 1 \times (-4)}}{2 \times 1}$$

$$= \frac{7 \pm \sqrt{49 - (-16)}}{2} = \frac{7 \pm \sqrt{65}}{2}$$

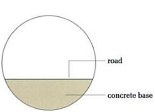
So the equation $x - 7 = \frac{4}{x}$ has two distinct solutions,

$$x = \frac{7 + \sqrt{65}}{2} \quad \text{and} \quad x = \frac{7 - \sqrt{65}}{2}.$$

Evaluate these results using a calculator. What do you notice? Both answers are irrational so the calculator screen should give decimal expansions that do not recur. The numerical values correct to two decimal places are 7.53 and -0.53.

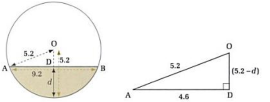
Example 9

It is proposed that a new tunnel be built under the English Channel. This tunnel will be for cars to drive through. The road will be built on a concrete base inside the circular tunnel.



If the radius of the tunnel is 5.2 m and the width of the road surface is to be 9.2 m, what depth of concrete should be used?

Solution



The first step is to redraw the diagram using the given measurements.

Notice that AD is half the width of the road surface, and that OD is $(5.2 - d)$ metres.

Let d metres be the depth of concrete. From the diagram identify the right-angled triangle with the road surface as base.

Using Pythagoras' theorem,

$$OA^2 = AD^2 + OD^2$$

$$5.2^2 = 4.6^2 + (5.2 - d)^2$$

$$5.2^2 = 4.6^2 + (5.2^2 - 10.4d + d^2)$$

$$\text{So } 0 = 4.6^2 - 10.4d + d^2$$

Multiply out the bracketed term.

Notice the 5.2^2 on both sides of the equation.

That is, $d^2 - 10.4d + 21.16 = 0$.

This is a quadratic equation in d where $a = 1$, $b = -10.4$ and $c = 21.16$. Now use the formula.

$$d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

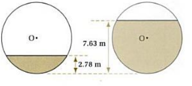
$$= \frac{10.4 \pm \sqrt{(-10.4)^2 - 4 \times 1 \times 21.16}}{2}$$

$$= \frac{10.4 \pm \sqrt{23.52}}{2}$$

$$= 5.2 \pm 2.425 \text{ (3 d.p.)}$$

Remember not to reduce working to the required number of decimal places until the final solution is written.

Notice that there are two solutions. The first, 2.78 m, has the road in the lower half of the tunnel. The second, 7.63 m, has the road in the upper half of the tunnel.



This example illustrates one use of the quadratic formula in a problem solving context.

Remember that although in many applications both solutions can be interpreted in the context of the problem, one solution will usually be preferable. Notice also that this quadratic equation had decimals as coefficients. The quadratic formula has given solutions that have been rounded, to give answers correct to three significant figures.

Graphical methods

A graphical calculator can be used to solve quadratic equations. This is also a good method to use if you simply want to check solutions from factorisation, completing the square or the quadratic formula.

Example 10

Using a graphical calculator, or graph plotting software on a computer, draw the graphs of the following.

a $y = x^2 - 4x - 5$ **b** $y = x^2 - 3x$
c $y = x^2 - 6x - 5$ **d** $y = 3x^2 + 4x - 5$

Trace
The TRACE facility or graphical calculator allows a point to move along the last graph drawn, simultaneousl showing either the x-coordinate or the y-coordinate of the point.

Using the trace facility, find the coordinates of the points of intersection with the x axis. Now compare these results with the results from Examples **2a**, **2c**, **4** and **6**. What do you notice? You should find that the calculator gives either an exact answer or a decimal approximation to the solutions calculated by other methods. Remember that the accuracy of these solutions will depend upon your calculator. Many will not state irrational results exactly; instead a decimal is given to 8, 10 or 12 places.

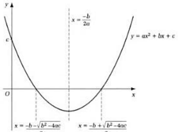
Notice that the graphs of these quadratic equations all have the same basic shape. This curve is known as a **parabola**, but can be transformed by changing the values of the coefficients a, b and c in the expression $ax^2 + bx + c$.

The equation $y = ax^2 + bx + c$ cross the x-axis at $y = 0$.

Notice also that all the graphs are symmetrical. Is this line of symmetry related to the coefficients a, b and c? Think back to the quadratic formula. This gives the solutions to the equation $ax^2 + bx + c = 0$ in a form that helps answer this question.

The points of intersection with the x-axis are written as $(\frac{-b}{2a} \pm$ a square-root term, $0)$. This suggests that the **line of symmetry** for the quadratic is $x = -\frac{b}{2a}$.

What about the square-root term? What does $\sqrt{b^2 - 4ac}$ represent? The expression $b^2 - 4ac$ is known as the **discriminant**. It can be used to give an indication of how many times the graph will cross the x-axis, as follows.



- If $b^2 - 4ac > 0$, it has two real square roots and $ax^2 + bx + c = 0$ has two distinct solutions: the graph will cross the x-axis twice – at $(\frac{-b+\sqrt{b^2-4ac}}{2a}, 0)$ and $(\frac{-b-\sqrt{b^2-4ac}}{2a}, 0)$.
- If $b^2 - 4ac = 0$, $ax^2 + bx + c = 0$ has one (repeated) solution: the graph will touch the x-axis at $(-\frac{b}{2a}, 0)$.
- If $b^2 - 4ac < 0$, $ax^2 + bx + c = 0$ has no (real) solutions: the graph will not cross the x-axis.

Example 11

Write down the equation of the line of symmetry of the graphs of the following quadratics, and predict the number of times the graph will cross the x-axis.

a $y = x^2 + x + 3$ **b** $y = x^2 + 5x + 6$ **c** $y = x^2 + 2x + 1$

Solution

a $y = x^2 + x + 3$

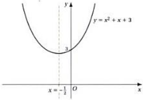
When $y = 0$ (on the x-axis), $x^2 + x + 3 = 0$, and $a = 1$, $b = 1$ and $c = 3$ in the quadratic formula.

The line of symmetry, $x = -\frac{b}{2a}$, is $x = -\frac{1}{2}$.

To check to see if (and how many times) the graph crosses the x-axis, check the discriminant.

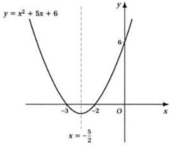
$b^2 - 4ac = 1^2 - (4 \times 1 \times 3) = 1 - 12 = -11 < 0$, so the graph doesn't cross the x-axis.

Draw the graph of each quadratic.

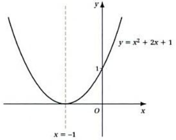


b $y = x^2 + 5x + 6$
When $y = 0$, $a = 1$, $b = 5$ and $c = 6$ in the quadratic formula.
The line of symmetry, $x = -\frac{b}{2a}$, is $x = -\frac{5}{2}$.
The discriminant $b^2 - 4ac = 5^2 - (4 \times 1 \times 6) = 25 - 24 = 1 > 0$, so the graph will cross the x-axis at two places.

Check that the graph cuts the x-axis at $x =$ and $x = -2$ using factorisation or the quadratic formula.



c $y = x^2 + 2x + 1$
Here $a = 1$, $b = 2$ and $c = 1$. The line of symmetry, $x = -\frac{b}{2a}$, is $x = -\frac{2}{2}$. That is, $x = -1$.
The discriminant $b^2 - 4ac = 2^2 - (4 \times 1 \times 1) = 4 - 4 = 0$, so the graph will touch the x-axis at one point. Check that $x^2 + 2x + 1 = 0$ when $x = -1$ by factorisation (so the graph touches at $(-1, 0)$).



1.5 Solving Quadratic Equations

Exercise

Technique

1 Solve the following:

a $(x-3)(x+2)=0$
b $(x-3)(x-4)=0$
c $(x-1)(x+3)=0$
d $(x+5)(x+1)=0$
e $(2x+5)(2x+5)=0$
f $x(x+2)=0$
g $12(x+3)(2x+1)=0$
h $3(x-7)(3x+4)=0$

2 Solve, by factorising, the following equations:

a $x^2+5x+4=0$
b $x^2+2x-15=0$
c $x^2-6x+5=0$
d $x^2-x-6=0$
e $4x^2-19x+12=0$
f $9x^2+24x+16=0$

3 Solve, by completing the square, the following equations:

a $x^2-6x-16=0$
b $x^2+2x-8=0$
c $x^2-2x-3=0$
d $x^2-6x+1=0$
e $2x^2-2x-1=0$
f $-3x^2+8x+7=0$

3 a, e

4 Solve, using the quadratic formula, the following equations:

a $3x^2-2x-8=0$
b $3x^2+10x-8=0$
c $2x^2+x-4=0$
d $x^2-12x-5=0$
e $2x^2+15x+6=0$
f $3x^2-18x+10=0$

5 Solve the following equations. In each case check the solutions by using a graphical calculator to find the points of intersection between the quadratic and the x-axis:

a $8x^2-24x+6=0$
b $3x(x-4)+5=-6$
c $3(x^2-2)=2(9x-2)$
d $x^2+6x+4=0$
e $2x^2+6x+2=0$
f $\dfrac{x^2+3}{3x}=2$

6 Write the following expressions in the form $a(x+p)^2+q$:

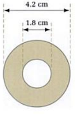
a x^2-2x+3
b x^2+4x+1
c $-x^2+2x+2$
d $-x^2+8x-19$
e $-2x^2+5x-3$
f $2x^2-3x-2$

6 b, e

Contextual

1 The formula $h = ut - \frac{1}{2}gt^2$ gives the height h a body will reach after time t, when it is thrown vertically upwards with velocity u, where g is a constant. Calculate t when $g = 9.8$, $u = 16$ and $h = 6$. Why are there two answers?

2 What is the shaded area of the washer illustrated here, where the diameter of the washer is 4.2 cm and the diameter of the hole is 1.8 cm?

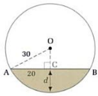


3 The sum of the first n natural numbers $(1 + 2 + 3 + \ldots + n)$ is given by the formula $S = \frac{1}{2}n(n + 1)$. If the sum of the numbers is 78, how many numbers have been added?

4 The formula $\frac{1}{2}n(n - 3)$ defines the number of diagonals in a polygon where n is the number of sides. A chef cuts a cake along its 65 diagonals. How many sides does the cake have?

5 By completing the square, find the minimum value of $3x^2 - 12x + 13$.

6 How deep is the water in this oil drum, given that the radius is 30 cm and AC is 20 cm?



Hint: Use Pythagoras' theorem in triangle OCA, and let $OC = 30 - d$.

1.6 Simultaneous Equations

Polynomial equations can be graphed. If two or more polynomials are graphed on the same axes then the graphs may cross. The coordinates of the point (or points) where the graphs cross satisfy both polynomial equations at the same time; that is, simultaneously.

Simultaneous linear equations

Linear equations may be put in the form $y = a_1x + a_0$ where a_1 and a_0 are real numbers. The highest power of the variable x is 1, so they are of degree 1. When graphed, these equations produce straight lines, which is why they are called linear.

$x + y = 3$ is linear and can be written as $y = -x + 3$.

A system of two linear equations can sometimes be solved simultaneously, using:

- substitution
- elimination
- graphical methods.

Two equations can be solved simultaneously, if a solution (or solutions) can be found that satisfies both equations.

Example 1

Solve the equations $x + 2y = 7$ and $2x + 3y = 10$ simultaneously.

Solution

1. Using the **substitution** technique gives a solution as follows.

$x + 2y = 7$ call this equation [1]

$2x + 3y = 10$ call this equation [2]

Make x the subject of equation [1]. ◄ ① **Make one variable (letter) the subject of one of the equations.**

$$x = 7 - 2y$$

That is, write the equation in the form $x = \ldots$ or $y = \ldots$

Substitute for x in equation [2]. ◄ ② **Substitute.**

$2x + 3y = 10$, so $2(7 - 2y) + 3y = 10$ call this equation [3]

Equation [3] now has only one variable (letter). Solve this (find the variable) by multiplying out the bracket and collecting like terms.

$$14 - 4y + 3y = 10$$

◄ ③ **Solve the new equation.**

$$14 - y = 10$$
$$14 - 10 = y$$
$$4 = y$$

Substitute this value of y back into equation [1].

$$x = 7 - 2y$$
$$x = 7 - (2 \times 4) = 7 - 8$$
$$x = -1$$

◄ ④ **Substitute this value into the first equation to find the value of the other variable.**

The values of x and y that simultaneously satisfy both equations are $x = -1$ and $y = 4$. How can we check this? Substitute the values back into the original equations.

$-1 + (2 \times 4) = 7$ and $(2 \times -1) + (3 \times 4) = 10$

◀ ⑤ Check by substituting the values back into th original equations.

Since both of these numeric equations are valid, $x = -1$ and $y = 4$ are the simultaneous solutions to $x + 2y = 7$ and $2x + 3y = 10$.

2. Using the **elimination** technique gives a solution of the following form.

$x + 2y = 7$ call this equation [1]

$2x + 3y = 10$ call this equation [2]

We want to make the coefficient of x the same in each equation. Multiplying equation [1] by 2,

◀ ① Multiply each equation by a valu that will make the coefficient of one c the variables the same in both equations.

$2x + 4y = 14$ call this equation [3]

Notice that x now has a coefficient of 2, the same as in equation [2]. We can eliminate the terms in the x variable from equations [2] and [3] by subtracting corresponding terms in these equations.

$2x + 4y = 14$

$2x + 3y = 10$

◀ ② Combine the ne equations to eliminate the variable with the same coefficient (usually by subtraction), and solve the resulting equation.

Taking each term separately, $2x - 2x = 0$ (so the new combined equation has no x term), $4y - 3y = y$ (so the new combined equation simply has y on the left-hand side), and $14 - 10 = 4$ (so the new combined equation simply has 4 on the right-hand side).

So $y = 4$.

Why were the equations subtracted? We have made the number of 'x's the same, and any number minus itself always gives zero. So subtracting the equation eliminates the x terms from the calculation.

Now substitute $y = 4$ back into one of the original equations to find x.

◀ ③ Substitute this into one of the original equations find the value of th other variable.

$$x + (2 \times 4) = 7$$
$$x + 8 = 7$$
$$x = -1$$

As before, $x = -1$ and $y = 4$ are solutions.

◀ ④ Check by substituting both variables in the original equations.

We could have eliminated y instead of x in the first step, but both equations would have required multiplication because the coefficient of y in equation [1] is 2 and the coefficient in equation [2] is 3.

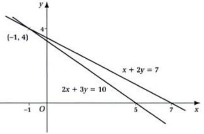
Equation [1] multiplied by 3 gives $3x + 6y = 21$

Equation [2] multiplied by 2 gives $4x + 6y = 20$

The coefficients of y are now the same, so that term could be eliminated by subtracting one equation from the other.

3. Using a **graphical method**, on the same axes draw the graphs of $x + 2y = 7$ and $2x + 3y = 10$. Notice that the two lines cross. The point where they cross is on both lines, and so satisfies both equations. Using a graphical calculator the 'trace' function can be used to find the coordinates of the point of intersection. In this example it is $(-1, 4)$, as before.

Solve these equations simultaneously and show that you get the same solution.



The graphical technique is a useful tool for checking the algebraic methods.

Simultaneous equations: linear and non-linear

A non-linear equation is a polynomial equation of degree 2 or higher. These can be solved using the same techniques as for linear equations. However, there is usually more than one solution.

Example 2

Solve the simultaneous equations $x - y + 3 = 0$ and $x^2 + y^2 = 29$.

Solution

Using the substitution technique, make y the subject in the first equation.

① Make one variable (letter) the subject of one of the equations.

$$x - y + 3 = 0 \Rightarrow y = x + 3$$

Now substitute for y in the second equation.

$$x^2 + (x+3)^2 = 29$$
$$x^2 + (x^2 + 6x + 9) = 29$$
$$2x^2 + 6x - 20 = 0$$
$$2(x^2 + 3x - 10) = 0$$

◄ ② **Substitute this result into the second equation.**

Factorising the quadratic,

$2(x+5)(x-2) = 0$

Since $2 \neq 0$ then $(x+5) = 0$ or $(x-2) = 0$.

Therefore, $x = -5$ or $x = 2$.

◄ ③ **Solve the new equation in one variable.**

Notice that there are two solutions for x. Substitute each of these into the equation where y is the subject.

$y = x + 3$

$y = -5 + 3$ or $y = 2 + 3$

$y = -2$ or $y = 5$

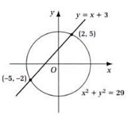
◄ ④ **Substitute the new-found values of x into the first equation.**

So there are two distinct solutions: $x = -5$, $y = -2$ and $x = 2$, $y = 5$. These can be written in coordinate form as $(-5, -2)$ and $(2, 5)$. Check these solutions graphically.

If you are using a graphical calculator it may not draw $x^2 + y^2 = 29$. Instead, rearrange the equation to make y the subject.

$$x^2 + y^2 = 29 \Rightarrow y^2 = 29 - x^2 \Rightarrow y = \pm\sqrt{29 - x^2}$$

If you 'overlap' $y = +\sqrt{29 - x^2}$ and $y = -\sqrt{29 - x^2}$, your calculator should produce a circle, centred on the origin. Set the range as $x_{\min} = -6$, $x_{\max} = 6$, $y_{\min} = -6$ and $y_{\max} = 6$, and use the trace function to find the points of intersection with the line $y = x + 3$. What do you notice about the solutions? The straight line crosses the circle at the points $(-5, -2)$ and $(2, 5)$.



If your calculator produces an ellipse on screen you may need t choose a more appropriate scale to make the shape circul

1.6 Simultaneous Equations

Exercise

Technique

1 Solve the following pairs of linear simultaneous equations:

a $2x + y = 8$ and $3x + 2y = 14$
b $x + 3y = 11$ and $5x + y = 13$
c $x + 2y = 13$ and $4x - 5y = -13$
d $3x + y = 11$ and $2y - 5x = 11$
e $2x + 3y = 7$ and $3x - y = -6$
f $4x - y = -0.5$ and $3x + 2y = -4.5$

1 a, f

2 Solve the following pairs of simultaneous equations:

a $y = 2(x - 2)$ and $y = x^2 - 3x + 2$
b $y = x^2 - 2x - 1$ and $y = x - 3$
c $y - 5x = 2$ and $y = x^2 + 5x - 2$
d $y = 15 - x$ and $y = x^2 - 2x + 3$
e $x - y = 2$ and $x^2 + y^2 = 34$
f $x - y = 3$ and $x^2 + xy + 2y^2 = 22$

2 a, e

3 Solve the following pairs of simultaneous equations:

a $70 - T = 7a$ and $T - 40 = 5a$
b $\frac{1}{3}(2x + y) = 4$ and $\frac{1}{5}(13x - 4y) = 3$
c $2y - 2x + 1 = 0$ and $x^2 - xy + 2y^2 = 8$
d $2x = 2y - 9$ and $y = \frac{8}{x}$

Contextual

1 The graph $y = x^2 + 2x - 3$ crosses the line $y = 4x$ at the points A and B. Find the coordinates of the points.

2 For a football match, the attendance was 44,000 people: x people paid £30, y people paid £20 and the total receipts for the game came to £1.2 million. How many people paid for the higher price tickets?

3 Four CDs and three tapes cost £126. Two CDs and five tapes cost £112. Find the individual costs of a CD and a tape.

4 Melanie's straight line passes through the points (2, 7) and (5, 13). Using the general equation for a straight line, $y = mx + c$, find m and c.

5 Three years from now, Callum will be twice as old as Lydia was five years ago. At the moment, half their combined ages is 16. Find their ages.

Consolidation

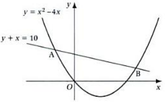
Exercise A

1 Diane throws a tennis ball vertically upwards. Given the formula

$$h = ut + \tfrac{1}{2}gt^2$$

where h is the height, u is the initial velocity of the ball and g is the gravitational acceleration, calculate the time for the ball to reach a 6 m height when $u = 14\,\text{m}\,\text{s}^{-1}$ and $g = -9.8\,\text{m}\,\text{s}^{-2}$.

2 The diagram shows the graphs of $y = x^2 - 4x$ and $y + x = 10$, which cross at A and B. Find the coordinates of A and B.

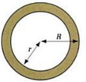


3 An opera is attended by 240 people: x people paid £31, y people paid £16, and the box office took in £5595.

a Form two equations using this information.

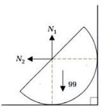
b Solve them to find out how many people paid £31.

4 A cement company supplies cement for 1200 m of underground concrete tunnels. Show that the area of the cross-section of the tunnel shown is $\pi(R - r)(R + r)$. If $R = 1$ m and $r = 0.95$ m, find the volume of concrete mix needed to make the tunnel in terms of π.



5 A ball is kicked, and just lands on a roof 5 m high. Using some basic knowledge to find the angle it was kicked at, θ, Ahmed correctly comes up with the formula $2 = 4\tan\theta - (1 + \tan^2\theta)$. Solve this equation to find the angle the ball was kicked at.

6 A uniform solid hemisphere is at rest. Kate looks at the forces involved and deduces that $\frac{3}{4}N_1 - N_2 = 0$ and $N_1 + \frac{1}{2}N_2 - 99 = 0$. Use her equations to find N_1 and N_2.



7 Show that the elimination of x from the simultaneous equations $x - 2y = 1$ and $3xy - y^2 = 8$ produces the equation $5y^2 + 3y - 8 = 0$. Solve this quadratic equation and hence find the pairs (x, y) for which the simultaneous equations are satisfied.

(ULEAC)

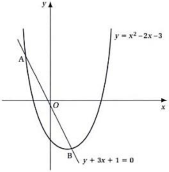
Exercise B

1 Julie throws a cricket ball vertically upwards. Given the formula

$$h = ut + \tfrac{1}{2}gt^2$$

where h is the height, u is the initial velocity of the ball and g is the gravitational acceleration, calculate the time for the ball to reach a height of 13 m when $u = 18\,\text{m s}^{-1}$ and $g = -9.8\,\text{m s}^{-2}$.

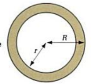
2 The graphs $y = x^2 - 2x - 3$ and $y + 3x + 1 = 0$ cross at A and B. Find the coordinates of A and B.



3 A 'Science Fiction mania-day' is attended by 600 people: x people pay £22 at the door, y people pay £17 for tickets in advance, and the organisers took a total of £11,400.

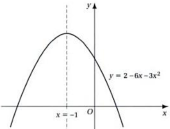
a Form two equations using this information.
b How many paid at the door and how many bought tickets in advance?

4 A plastics company supply 800 m of underground cable tubing. Show that the area of the cross-section is $\pi(R-r)(R+r)$. If $R = 5.2$ cm and $r = 4.8$ cm, find the volume of plastic needed to make the tubing.



5 Elaine hits a golf ball over a tree. Using some basic knowledge to find the angle at which the ball was initially hit, her caddy correctly comes up with the formula $5 = 5\tan\theta - (1 + \tan^2\theta)$. Solve the equation to find what the angle might be.

6 The sketch shows the curve with equation $y = 2 - 6x - 3x^2$, and its axis of symmetry, $x = -1$.



a Give the coordinates of the vertex and the value of y when $x = 0$.
b Find the values of the constants a and b such that
$2 - 6x - 3x^2 = a(x+1)^2 + b$.

(OCSEB)

7 Solve the following quadratic equations:

a $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 4 = 0$
b $2(2^x)^2 - 3(2^x) + 1 = 0$

Hint: In **a** put $y = x^{\frac{1}{3}}$
in **b** put $y = 2^x$.

Applications and Activities

1 Complete the square for the following six quadratics:

a $y = x^2 - 6x - 16$

b $y = x^2 + 2x - 8$

c $y = x^2 - 2x - 3$

d $y = 2x^2 - 2x - 1$

e $y = x^2 - 6x + 1$

f $y = -3x^2 + 8x + 7$

2 Now draw their graphs using graph paper or a graphical calculator. Look at the coordinates of the bottom (or top) of your curve. What do you notice when you compare these coordinates to the equation in its 'complete the square' form?

Summary

- Surds are irrational numbers containing a square root, and have the following properties:

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$
$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$
$$a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}$$
$$a\sqrt{b} - c\sqrt{b} = (a - c)\sqrt{b}$$

- To rationalise a surd denominator, multiply by the conjugate:

$$\frac{1}{a + \sqrt{b}} = \frac{1}{a + \sqrt{b}} \times \frac{a - \sqrt{b}}{a - \sqrt{b}} = \frac{a - \sqrt{b}}{a^2 - b}$$

- The properties of indices are:

$$a^p \times a^q = a^{p+q}$$
$$(a^p)^q = a^{p \times q}$$
$$a^p \div a^q = a^{p-q}$$
$$a^0 = 1 \text{ provided } a \neq 0$$
$$a^{-p} = \frac{1}{a^p}$$
$$a^{\frac{1}{p}} = \sqrt[p]{a}$$
$$a^{\frac{p}{q}} = \sqrt[q]{a^p} = (a^{\frac{1}{q}})^p$$

- An equation in the form $y = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, is a **polynomial equation.**
- An equation of the form $y = ax + b$ has degree (or order) 1 and is called a **linear equation.**
- A polynomial equation of degree (or order) 2 is called a **quadratic equation** and is of the form $y = ax^2 + bx + c$;
- In $ax^2 + bx + c$, the **coefficient** of x is b, and c is the **constant** term.
- When factorising, common factors are extracted; $ax + bx = x(a + b)$.
- A **difference of two squares** (square minus a square) is factorised according to the rule

$$a^2 - b^2 = (a - b)(a + b)$$

- Complete the square when factorising by adding the square of half the coefficient of x. By comparing coefficients, you can then write a quadratic in the form $a(x + p)^2 + q$.
- The formula for solving $ax^2 + bx + c = 0$ is

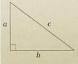
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The **discriminant**, $b^2 - 4ac$, informs you of the behaviour of the graph of the quadratic.
- Simultaneous equations can be solved by substitution, elimination and graphical methods.

2 Coordinate Geometry

What you need to know

- How to change the subject of an equation.
- How to use Pythagoras' theorem; $a^2 + b^2 = c^2$.
- How to expand $(a+b)^2$.
- How to expand $(a-b)^2$.



$a^2 + b^2 = c^2$

Review

1 Make y the subject of the following equations:

a $y - x = 7$

b $y + x + 6 = 11$

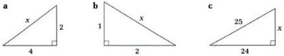
c $y + 4 = 2(x + 3)$

d $y - 5 = 3(x - 2)$

e $\dfrac{y+2}{x-3} = 4$

f $\dfrac{y-3}{x+2} = \frac{1}{2}$

2 For each triangle **a–c**, find the length of the lettered side without using a calculator.



3 Expand the following expressions:

a $(x+2)^2$

b $(x+3)^2$

c $(x+5)^2$

d $(a+b)^2$

e $(2x+1)^2$

f $(3x+2)^2$

4 Expand the following expressions:

a $(x-3)^2$

b $(x-1)^2$

c $(a-b)^2$

d $(3x-2)^2$

e $(2x-3)^2$

f $(5x-1)^2$

2.1 Coordinate Geometry

Coordinate geometry is the study of straight lines and curves using algebraic methods. The **Cartesian coordinate system** (named after Descartes) is one where axes are drawn perpendicular to each other and the same scale is chosen on each axis. If two points are plotted on this set of axes they can always be joined by a single straight line.

René Descartes (1596–1650)
Descartes linked geometry and algeb
In his work *Discour de la Méthode* (163
he explained the principles of analyt geometry.

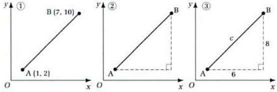
Example 1

If A is the point (1, 2) and B is the point (7, 10) what is the shortest distance between them?

Pierre de Ferma (1601–1665)
Fermat also worke out the method fo analytic geometry and published his own work in 1679

Solution

The shortest distance between the points is the length of the straight line joining them. By drawing this line, and then creating a right-angled triangle, Pythagoras' theorem can be applied.



Step ① Join A and B.
Step ② Draw lines parallel to the axes from A and B to create a right-angled triangle.
Step ③ Use the coordinates of A and B to find the lengths of the shorter sides.
Step ④ State Pythagoras' theorem, and use it to find c, the length of the line joining A and B.

$$c^2 = a^2 + b^2$$
$$= 6^2 + 8^2$$
$$= 36 + 64 = 100$$
$$\text{So } c = \sqrt{100}$$
$$c = 10$$

Notice that a crucial step is to find the lengths of the shorter sides from the coordinates of A and B. This is done by finding the difference between the

x-coordinates of A and B, and the difference between the y-coordinates of A and B. So in Example 1, for A (1, 2) and B (7, 10),

length parallel to x-axis = x-coordinate of B − x-coordinate of A

$= 7 - 1 = 6$

length parallel to y-axis = y-coordinate of B − y-coordinate of A

$= 10 - 2 = 8$

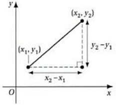
So we found the shortest distance between A (1, 2) and B (7, 10) by using

$$c = \sqrt{a^2 + b^2} = \sqrt{(7-1)^2 + (10-2)^2}$$

◀ **Check this using your calculator.**

In general the distance between two points A and B with coordinates (x_1, y_1) and (x_2, y_2) respectively is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



This process is still valid when $x_2 - x_1$ and $y_2 - y_1$ are negative.

◀ Learn this result.

Example 2

Find the distance between the following pairs of points:

a A (2, 7) and B (1, 9)
b C (−3, 7) and D (−2, −1)
c E (1, −2) and F (t, t^2).

Solution

a Distance $\text{AB} = \sqrt{(1-2)^2 + (9-7)^2}$

$= \sqrt{(-1)^2 + 2^2} = \sqrt{1+4}$

$= \sqrt{5}$

Remember that you can leave the answer in surd form.

b Distance $\text{CD} = \sqrt{[(-2)-(-3)]^2 + [(-1)-7]^2}$

$= \sqrt{(-2+3)^2 + (-1-7)^2}$

$= \sqrt{1^2 + (-8)^2} = \sqrt{1+64}$

$= \sqrt{65}$

c Distance EF $= \sqrt{(t-1)^2 + [t^2 - (-2)]^2}$

$= \sqrt{(t-1)^2 + (t^2+2)^2}$

Expand each bracket inside the square root.

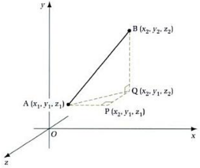
$= \sqrt{t^2 - 2t + 1 + t^4 + 4t^2 + 4}$

Collect like terms.

$= \sqrt{t^4 + 5t^2 - 2t + 5}$

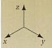
The formula for the distance between two points can be extended into three dimensions (3D). In the Cartesian system there would now be three axes (x, y and z) and points would have three coordinates, (x, y, z). Consider the points A (x_1, y_1, z_1) and B (x_2, y_2, z_2). Draw a sketch to show these points in 3D space. What is the distance between A and B now?

The x, y and z coordinate axes are also called Ox, Oy and Oz.



Axes x, y and z are perpendicular. In this view the z-axis is drawn coming 'out of' the page.

In mechanics it is more usual to make the z-axis vertical.



Construct two triangles as indicated.

By Pythagoras' theorem.

In ΔAPQ, $(AQ)^2 = (AP)^2 + (PQ)^2$ call this equation [1]

In ΔAQB, $(AB)^2 = (AQ)^2 + (BQ)^2$ call this equation [2]

Substitute for $(AQ)^2$ from equation [1] into equation [2]

$$(AB)^2 = (AP)^2 + (PQ)^2 + (BQ)^2$$

But $AP = x_2 - x_1$, $BQ = y_2 - y_1$ and $PQ = z_2 - z_1$

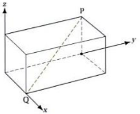
so $(AB)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

The distance $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

This result is still true if $x_2 - x_1$, $y_2 - y_1$ or $z_2 - z_1$ is negative.

Example 3

An infra-red alarm detector placed in the corner of the grand hall, at P (0, 12, 5), has a range of 14 m. Will it be able to detect a burglar entering the room at Q (4, 0, 0)?



Solution

The detector will be able to detect a burglar if the distance between P and Q is less than 14 m.

$$\text{Distance PQ} = \sqrt{(0-4)^2 + (12-0)^2 + (5-0)^2}$$

$$= \sqrt{(-4)^2 + 12^2 + 5^2}$$

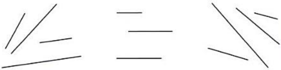
$$= \sqrt{16 + 144 + 25} = \sqrt{185}$$

$$\text{Distance PQ} = 13.6\text{ m}$$

Since this is within the 14 m range the burglar will be detected.

The gradient of a line joining two points

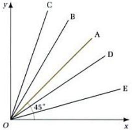
The gradient of a line is a measure of its steepness. It is given by the ratio of the change in the y-coordinate to the change in the x-coordinate. Gradients can be positive, zero or negative.



Positive gradients (rising) | Zero gradients (horizontal) | Negative gradients (falling)

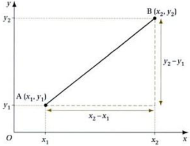
Consider more closely some positive gradients. The gradient of line A is 1. Notice how it makes an angle of 45° with the x-axis. The gradient of lines

B and C are greater than 1. These lines are steeper than line A. The gradients of lines D and E are less than 1, but bigger than 0. Which line has the smallest gradient? The gradient of line E is smallest, because it is closest to the horizontal.

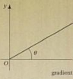


The gradient can be calculated algebraically using the rule

$$\text{gradient} = \frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$



An alternative is to think of the gradient the tangent of the an[gle] between the line and the x-axis.



The gradient of the straight line joining A (x_1, y_1) and B (x_2, y_2) is given by gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

◀ **Learn this result.**

This result gives both the sign (+/−) and th[e] magnitude (size) of th[e] gradient.

Example 4

a Find the gradient of the straight line joining A (1, 2) and B (7, 10).

b If the gradient of the straight line joining P $(a, 3)$ and Q (2, 8) is 5, find the value of a.

Solution

a gradient $= \dfrac{10-2}{7-1} = \frac{8}{6} = \frac{4}{3}$

The gradient is often represented mathematically by the letter m. It could be written here as gradient $m_{AB} = \frac{4}{3}$ or $m = \frac{4}{3}$.

b We know that gradient $m_{PQ} = \dfrac{y_2 - y_1}{x_2 - x_1}$

So $\quad 5 = \dfrac{8-3}{2-a}$

Then $\quad 5(2-a) = 8 - 3 = 5$

So $\quad 2 - a = 1$

$$-a = 1 - 2 = -1$$

$$a = 1$$

Multiply both sides by $(2-a)$.
Divide both sides by 5.

Parallel lines

Use a graphical calculator to draw the graphs of $y = 3x$, $y = 3x + 5$ and $y = 3x - 1$. What do you notice? The lines are parallel. Now calculate the gradient of each line. The gradient of each line is 3, the same as the coefficient of x in each equation.

Try drawing some graphs of your own linear equations where the coefficient of x is the same. What happens? If the equation starts $y =$, then when the coefficient of x is the same the equations produce lines that are parallel. These lines never cross and so the equations that represent them cannot be solved simultaneously. Conversely, if two linear equations cannot be solved simultaneously then their graphs must be parallel lines.

Example 5

Show that the following pairs of lines are parallel: $y = 2x + 3$, $y = 2x + 5$.

Solution

$y = 2x + 3$ call this equation [1]

$y = 2x + 5$ call this equation [2]

Recall how to solve two linear equations simultaneously.

$2x + 3 = 2x + 5$ ◀ **Substitute for y from equation [1] in equation [2].**

$2x - 2x = 5 - 3$

$x(2 - 2) = 5 - 3 = 2$

$x = \frac{2}{0}$

Take out the common factor and rearrange the equation.

Division by zero doesn't give a real number; it is undefined. So x cannot be found to satisfy both equations simultaneously. This means the lines do not cross; so they are parallel.

It is often quicker to show that two straight lines are parallel by comparing the coefficients of x. To do this, we sometimes need to rearrange the equations in order to make y the subject.

Example 6

Show that the following pairs of lines are parallel: $y = 2(3x + 1)$, $2y - 12x + 6 = 0$.

Solution

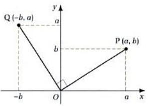
$$y = 2(3x + 1) \Rightarrow y = 6x + 2$$

$$2y - 12x + 6 = 0 \Rightarrow y = 6x - 3$$

The coefficient of x is the same in both equations, 6, so the lines are parallel.

Perpendicular lines

Consider a line OP where P is some point (a, b) and O is the origin. Rotate OP 90° anticlockwise about O (a quarter turn) and call this new line OQ. Now the angle between the lines OP and OQ is 90°. We say that OP is **perpendicular** to OQ; 'perpendicular to' means 'at right angles to'.



What do you notice about the coordinates of Q? Using the symmetry of the diagram, notice that the coordinates of Q are $(-b, a)$. These have the same numerical values, but there is a change of order and one change of sign in the x-coordinate.

Now consider the gradient of each line.

$$\text{gradient OP} = \frac{b}{a} \qquad \text{gradient OQ} = \frac{-a}{b}$$

Multiply these two gradients together. What happens? Their product is -1. In fact, the product of the gradients of perpendicular lines is always -1. This is a very useful test for whether two straight lines are perpendicular to each other.

The gradient of OQ c
found using the form
with $(x_2, y_2) = (0, 0)$.
gradient $= \frac{(0-a)}{[0-(-b)]}$
$= -\frac{a}{b}$

Example 7

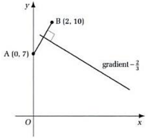
a Find the gradient of the line joining A (0, 7) and B (2, 10).

b Find the gradient of a line perpendicular to AB.

Solution

a Gradient of the line AB, $m_{AB} = \dfrac{10-7}{2-0} = \frac{3}{2}$

b Gradient of a line perpendicular to AB $= -\frac{2}{3}$, because $\frac{3}{2} \times (-\frac{2}{3}) = -1$.



Example 8

If A is the point (3, −2) and B is the point (5, 2), find the gradient of:

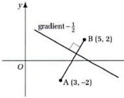
a the line AB

b a line perpendicular to AB.

Solution

a Gradient of the line AB, $m_{AB} = \dfrac{2-(-2)}{5-3} = \dfrac{4}{2} = 2$

b Gradient of a line perpendicular to AB is $-\frac{1}{2}$



2.1 Coordinate Geometry
Exercise
Technique

1 Find the shortest distance between the following pairs of points:

a (3, 5) and (1, 4)
b (3, 5) and (5, 6)
c (1, 7) and (−2, 3)
d (1, 7) and (0, −1)
e (−2, −3) and (−7, −1)
f (8, −4) and (−7, −4)
g (2, 1, 5) and (5, 13, 9)
h (7, −2, 18) and (−3, 3, 8)

2 Find the length of the straight line joining the following pairs of points:

a A (7, −1) and B (−2, 5)
b A (−1, −2) and B (−2, −5)
c A (3, 2) and B (−1, −5)
d A (−2, −1) and B (0, 3)
e A (4, −8) and B (0, 6)
f A (4, 4) and B (−1, 2)
g A (6, 11, −3) and B (1, 1, 7)
h A (2, 3, 4) and B (4, 6, 10)

3 Find the gradient of the straight line formed by joining the following pairs of points:

a (3, 2) and (5, 12)
b (2, 1) and (4, 9)
c (5, 3) and (7, 1)
d (0, 7) and (−2, 9)
e (−2, −1) and (6, −1)
f (3, 2) and (5, −8)

4 For the following pairs of points, A and B, find:

i the gradient of the line AB
ii the gradient of a line perpendicular to AB.

a A (0, 6) and B (2, 7)
b A (5, 2) and B (−3, −3)
c A (−3, 0) and B (2, −5)
d A (−3, 6) and B (−1, −3)
e A (−3, −2) and B (6, −6)
f A (−2, 0) and B (7, 2)

5 Rearrange the following equations to make y the subject. State whether the pairs of lines are parallel or perpendicular to each other.

a $y = 2x + 3$ and $y = 2x - 7$
b $y = 3x + 7$ and $y = 5 - \frac{1}{3}x$
c $y = 2x - 5$ and $y - 2x = 3$
d $2y + 6x + 8 = 0$ and $3x + y = -7$
e $8x + 2y = 6$ and $4y = 9 + x$
f $3y = 9(x - 1)$ and $6y + 2x = 6$

Contextual

1 Katie moves her position from the point (2, 6) to the point (5, 3) on the park map. The map is drawn to a scale of 1:10 000. Find

a the shortest distance in centimetres that Katie covers on the map
b the actual distance she moves in km.

2 Twins Peter and David radio their respective coordinates to each other. Peter is at position (3, 9) and David is at (−2, −3). How far apart are they?

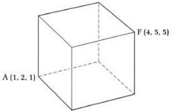
3 The night before military manoeuvres, an army troop are given starting coordinates, (965, 386), and finishing coordinates, (943, 379). By considering only the final two digits of each coordinate, find the distance on the Ordnance Survey map moved by the troop, in centimetres. If the scale of their map is 1:25 000, find the actual distance covered in kilometres.

4 According to a garden plan, the cottage (7, 9) is the same distance from the ash tree (5, 2) and the beech tree (9, 2). Investigate this statement.

5 A parallelogram is formed by lines joining the points P, Q, R and S. Given the coordinates of points P (−2, 3), Q (3, 4) and R (2, −1):

a Find the coordinates of S.
b Show that PQRS is a rhombus.

6 The diagram shows a sketch of a cuboid. Given the coordinates of A (1, 2, 1) and F (4, 5, 5), find the shortest distance between A and F.



2.2 The Equation of a Straight Line

Linear equations can be written in many forms. The general form is $ax + by + c = 0$ where a, b and c are real numbers.

Example 1

Rearrange the following equations into the general form $ax + by + c = 0$:

a $y = -3x - 8$

b $y + 2 = -\frac{1}{3}(x - 1)$

Solution

a $3x + y + 8 = 0$

b $y + 2 = -\frac{1}{3}(x - 1)$

$3y + 6 = -(x - 1)$

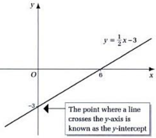
$3y + 6 = -x + 1$

$x + 3y + 5 = 0$

Multiply throughout b
3 to remove the fractio

The minus sign outsid
the bracket changes th
sign of each term insi
when the bracket is
removed.

The general equation can be rearranged to make y the subject. It is then written $y = mx + c$ When written in this form it instantly highlights two important properties; the gradient of the line and its intercept with the y-axis. Consider the line $y = \frac{1}{2}x - 3$. This is in the form $y = mx + c$, with $m = \frac{1}{2}$ and $c = -3$. Notice that the gradient is $\frac{1}{2}$ and the y-intercept is $(0, -3)$.



The equation $y = \frac{1}{2}x - 3$ can be written in other forms:

$$y = \frac{1}{2}x - 3$$
$$2y = x - 6$$
$$x - 2y - 6 = 0$$

Multiplying througho
by 2 removes the
fraction.

In the last version, the equation is in the more general form $ax + by + c = 0$, with $a = 1$, $b = -2$ and $c = -6$. Notice that $y = \frac{1}{2}x - 3$ and $x - 2y - 6 = 0$ both represent the same straight line.

Example 2

Find the gradient and y-intercept of the straight lines represented by the following equations and sketch their graphs:

a $3x + 3y - 7 = 0$ **b** $2x - 5y + 1 = 0$

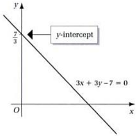
Solution

a $3x + 3y - 7 = 0 \Rightarrow 3y = -3x + 7$

$$y = \frac{-3x}{3} + \frac{7}{3} = -x + \frac{7}{3}$$

First express the equation in the form $y = mx + c$.

So the gradient is -1 and the y-intercept is $(0, \frac{7}{3})$.



◀ Check this sketch using a graphical calculator.

Graphical calculator support pack

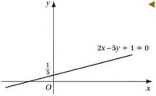
Remember to label the axes and origin on your sketch. Label the line with its equation, and mark the y-intercept.

b $2x - 5y + 1 = 0 \Rightarrow 2x + 1 = 5y$

$$\tfrac{2}{5}x + \tfrac{1}{5} = y$$

Remember to use the $y = mx + c$ form on your calculator.

So the gradient is $\frac{2}{5}$ and the y-intercept is $(0, \frac{1}{5})$.



◀ Check this sketch using a graphical calculator.

Graphical calculator support pack

The equations can be entered into the calculator in the form $y = \dots$.

The technique of rearranging equations into the form $y = mx + c$ is particularly useful when solving simultaneous equations with a graphical method.

Example 3

Solve the simultaneous equations $x + y = 4$ and $2y - 3x = 3$ using a graphical method.

Solution

We could rearrange each equation into the form $y = mx + c$ to identify the gradient and intercept for each graph. However, it is often quicker to draw each graph by calculating where they cross the axes.

To find where the lines cross the y-axis, put $x = 0$ into each equation. To find where the lines cross the x-axis, substitute $y = 0$ instead.

For $x + y = 4$, when $x = 0, y = 4$

and when $y = 0, x = 4$

This line crosses the axes at $(0, 4)$ and $(4, 0)$.

For $2y - 3x = 3$, when $x = 0$, $2y = 3$

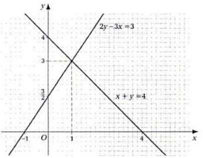
$$y = \tfrac{3}{2}$$

and when $y = 0, -3x = 3$

$$x = -1$$

This line crosses the axes at $(0, \tfrac{3}{2})$ and $(-1, 0)$.

The graphical solution of the simultaneous equations is given by the coordinates of the point of intersection of these two lines.



You should find that the coordinates of this point are $x = 1$ and $y = 3$. So the point $(1, 3)$ lies on both lines and $x = 1$, $y = 3$ is the solution because it satisfies both equations simultaneously.

An alternative metho is to rearrange each equation into the forn $y = mx + c$. Use a graphical calculator t draw these lines and find out where they cross.

Graphic calculat support pack

2.2 The Equation of a Straight Line

Exercise

Technique

1 Rearrange the following equations into the form $y = mx + c$:

a $3x + y + 7 = 0$
b $4x + y - 3 = 0$
c $\dfrac{y-2}{x-3} = 4$
d $\dfrac{y+2}{x-5} = 2$
e $x + y + 3 = 0$
f $2x - y - 5 = 0$
g $\dfrac{y-7}{2} = 4x$ 1 c
h $\dfrac{3-y}{2} = x$

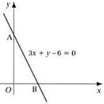
2 State the gradient and the y-intercept of the straight-line graphs produced by the following equations:

a $y = 5x - 3$
b $y = -2x + 3$
c $y = 7 - 2x$
d $y = \frac{1}{2}x + 5$

3 For the straight lines produced by following equations, find the gradient and the coordinates of the y-intercept:

a $2x + y + 8 = 0$
b $-2x + 3y - 2 = 0$
c $5x + 10y - 2 = 8$
d $2x - y + 7 = 4$
e $-3x + 7y = 14$ 3 e
f $ax + by + c = 0$

4 The equation of the line shown is given by $3x + y - 6 = 0$. Find the gradient and the coordinates of A and B.



5 Rearrange the equation $\dfrac{y-5}{x+4} = \frac{1}{2}$ into the form $y = mx + c$. Now sketch the graph of this equation.

6 Solve the simultaneous equations $x + y = 6$ and $4x - 2y + 6 = 0$ using a graphical method.

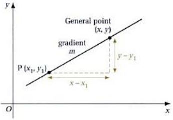
2.3 More on the Straight Line

Think about the information needed to describe a particular straight line. How can we write the equation of a line by looking only at the graph of the line? We find that we can write the equation if we know:

- the gradient of the line and the coordinates of a point on it
- the coordinates of two points on the line.

The equation of a line given its gradient and the coordinates of one point on the line

Suppose the gradient m and the coordinates of point P (x_1, y_1) on the line are known. A general point (x, y) on the line can then be used to find the equation of the line. The known gradient, m, can be expressed by the sides of a right-angled triangle drawn on P and the general point (x, y).



$$m = \frac{y - y_1}{x - x_1}$$

This can be rearranged into the very useful result,

$\mathbf{y - y_1 = m(x - x_1)}$ ◀ **Learn this result.**

Example 1

a Find the equation of the straight line with gradient 2 that passes through the point (3, 4).

b Find the equation of the straight line with gradient $\frac{1}{2}$ that passes through the point (−2, −6).

Solution

a Use $y - y_1 = m(x - x_1)$

Substitute the known values of m, x_1 and y_1

Then $y - 4 = 2(x - 3)$

$y - 4 = 2x - 6$

This equation can now be rearranged into either form of the equation of a straight line.

$ax + by + c = 0$, or $y = mx + c$.

$2x - y - 2 = 0$ or $y = 2x - 2$

b Use $y - y_1 = m(x - x_1)$

Then $y - (-6) = \frac{1}{2}(x - (-2))$

$$y + 6 = \tfrac{1}{2}(x + 2)$$

$$y + 6 = \tfrac{1}{2}x + 1$$

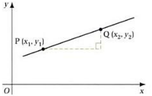
$$y = \tfrac{1}{2}x + 1 - 6 = \tfrac{1}{2}x - 5$$

Take care when manipulating the negative signs.

So the equation of the line is $y = \frac{1}{2}x - 5$.

The equation of a straight line passing through two known points

Suppose that two points P (x_1, y_1) and Q (x_2, y_2) are known to lie on the straight line. The gradient of this line can be found by drawing in a right-angled triangle.

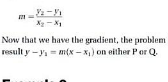


$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Now that we have the gradient, the problem is simple. We can use the result $y - y_1 = m(x - x_1)$ on either P or Q.

Example 2

Find the equation of the straight line that passes through the points A (2, 3) and B (4, 4).



Solution

The gradient of the straight line passing through A (2, 3) and B (4, 4) is

$$m = \frac{4-3}{4-2} = \frac{1}{2}$$

Now use $y - y_1 = m(x - x_1)$, with $m = \frac{1}{2}$ and B (4, 4) as the known point.

$$\begin{aligned} y - y_1 = m(x - x_1) \Rightarrow y - 4 &= \tfrac{1}{2}(x - 4) \\ y - 4 &= \tfrac{1}{2}x - 2 \\ y &= \tfrac{1}{2}x + 2 \end{aligned}$$

Check that you arrive at the same equation using A (2, 3) instead of B (4, 4).

The expression for the gradient, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, can be substituted directly into the equation for a straight line, $y - y_1 = m(x - x_1)$. At first the algebra might appear quite daunting, but it provides a very useful result.

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This equation can now be rearranged so that the x terms and y terms are separated, producing the equation of the straight line directly once the values of (x_1, y_1) and (x_2, y_2) are known.

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

◀ **Learn this result.**

Example 3

Find the equation of the straight line joining P (5, −6) and Q (−3, 2).

Solution

Using the result $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$

$$\frac{y - (-6)}{2 - (-6)} = \frac{x - 5}{(-3) - 5}$$

$$\frac{y + 6}{8} = \frac{x - 5}{-8}$$

Simplify the numeric components.

$$y + 6 = \frac{8}{-8}(x - 5)$$

$$y + 6 = -(x - 5) = -x + 5$$

$$y = -x + 5 - 6 = -x - 1$$

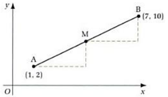
Make y the subject.

Check that this equation could also be written $x + y + 1 = 0$.

Finding the mid-point of a line

Given a line joining two known points the mid-point can be established using the mean of the x and y coordinates.

Recall that the mean average is the sum of the numbers divided by how many of them there are. In this case, taking the mean of two coordinates, we would add the coordinates and then divide by 2, to find the point half-way between them.



Consider the line joining A (1, 2) and B (7, 10). If M is the mid-point of this line then M is half-way between A and B, both horizontally and vertically. In this case the coordinates of M are (4, 6).

Notice that $(4, 6) = \left(\frac{1+7}{2}, \frac{2+10}{2}\right)$.

The mid-point M of a line joining the points A (x_1, y_1) and B (x_2, y_2) has coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Example 4

The vertices of an isosceles triangle are A (2, 7), B (5, 8) and C (4, 5).

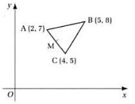
a State the coordinates of the mid-point of AC.

b Find the equation of the straight line through B and the mid-point of AC.

c Find the equation of the perpendicular bisector of BC.

The perpendicular bisector of a line is another line at right angles to the first, that passes through the mid-point of the original line.

Solution



a Let M be the mid-point of AC. Then the coordinates of M are $\left(\frac{2+4}{2}, \frac{7+5}{2}\right) = (3, 6)$.

2.3 More on the Straight Line

Exercise

Technique

1 Find the equation of the straight line with the given gradient passing through the stated point in each of the following:

a gradient 3, point (3, 2)
b gradient 6, point (−1, 2)
c gradient 5, point (3, −2)
d gradient −3, point (0, 4)
e gradient $\frac{1}{2}$, point (2, −3)
f gradient $-\frac{1}{3}$, point (−1, 4)

2 Find the equation of the straight line joining the following pairs of points:

a A (2, 4) and B (3, 6)
b R (−3, 4) and S (1, 2)
c T (−1, 1) and V (0, 6)
d A (0, −2) and B (3, 4)
e P (8, 6) and Q (2, 12)
f R (−1, −1) and S (5, 2)

3 Find the mid-point and the equation of the perpendicular bisector of AB in each of the following cases:

a A (1, −1) and B (3, 7)
b A (4, 1) and B (5, 0)
c A (−2, 5) and B (0, 3)
d A (−1, −2) and B (1, 6)

Contextual

1 A straight line passing through the points A (−1, 1) and B (p, 13), has gradient 2. Determine the value of p and find the equation of the straight line.

2 Consider two points, P (2, 7) and Q (4, 13).

a Find the mid-point of PQ.
b Find the gradient of PQ.
c Write down the gradient of the line perpendicular to the line PQ.
d Find the equation of the perpendicular bisector of PQ. Write it in the form $ax + by + c = 0$.

3 Sketch a diagram to show the points A (0, −1), B (4, 3) and C (4, 5). Let M be the mid-point of AB. Find the coordinates of M and write down the equation of the straight line that passes through M and C.

4 A is the point (6, 6) and B (8, 2) lies on the straight line $x - 2y - 4 = 0$.

a Find the equation of the straight line parallel to $x - 2y - 4 = 0$ that passes through A. Write it in the form $ax + by + c = 0$.
b Show that the straight line joining A and B is perpendicular to the line $x - 2y - 4 = 0$.
c Find the perpendicular distance between the two parallel lines.

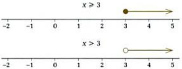
2.4 Inequalities

There are four inequality symbols.

$>$ means 'is greater than'
$\geq$ means 'is greater than or equal to'
$<$ means 'is less than'
$\leq$ means 'is less than or equal to'

Learn the mathematic
meaning of each of
these symbols.

Inequalities produce a range of acceptable answers. They can be represented on number lines using arrows. The base of the arrow is circular, and is shaded when the value is to be included in the range, and clear when the value is not to be included in the range.



Inequalities can be categorised into two types; those without variables (letters) and those with variables. Inequalities without variables are called **propositions**. Statements such as $2 < 7$, $\frac{1}{2} > \frac{1}{5}$, and $-3 < -2$ are propositions and are either true or false. Inequalities with variables can be solved using similar techniques to those used when solving equations. However, a solution set is often produced showing a range of acceptable answers.

A solution set is an
inequality, or a set of
inequalities, showing
the range of values th
are acceptable as
solutions to the
problem.

Example 1

Solve the inequalities:

a $2x + 1 \geq 7$ **b** $1 - 2x \geq 7$

c $5x + 2 < 3x + 10$ **d** $\dfrac{6 + x}{3} \geq x + 7$

Solution

a $2x + 1 \geq 7$

$2x \geq 6$ Subtract 1 from both
sides of the inequality

$x \geq 3$ Divide both sides by

b $1 - 2x \geq 7 \Rightarrow -2x \geq 6$

Notice that the coefficient of x is -2. To find x we need to divide both sides of the inequality by -2. When dividing by a negative number the inequality needs to be reversed.

Check this result by
substituting a value o
smaller than -3 in th
original inequality. D
the original inequalit
work?

$$-2x \geq 6 \Rightarrow x \leq \frac{6}{-2}$$

$$x \leq -3$$

c The process of solving linear inequalities is similar to solving linear equations. Collect the like terms together, with numbers on one side of the inequality and variables on the other.

$$5x + 2 < 3x + 10$$
$$\Rightarrow \quad 5x - 3x < 10 - 2$$
$$\Rightarrow \quad 2x < 8$$
$$\Rightarrow \quad x < 4$$

Check this result by substituting a value of x smaller than 4 in the original inequality.

d Again use a similar process to that used when solving linear equations. Eliminate the fractions (by multiplying by a common multiple), collect like terms and find a condition on x by using division.

$$\frac{6+x}{3} \geq x + 7$$
$$\Rightarrow \quad 6 + x \geq 3(x + 7)$$
$$\Rightarrow \quad 6 + x \geq 3x + 21$$
$$\Rightarrow \quad 6 - 21 \geq 3x - x$$
$$\Rightarrow \quad -15 \geq 2x$$
$$\Rightarrow \quad -\tfrac{15}{2} \geq x$$

That is, $x \leq -\frac{15}{2}$

This illustrates a useful 'trick'. The variable x is collected on the RHS of the inequality, making the resulting term, $2x$, positive.

Notice how the following rules were used:

- Any term can be added to, or subtracted from, both sides of the inequality and the symbol doesn't change.
- Both sides of an inequality can be multiplied, or divided, by the same positive number and the symbol doesn't change.
- When both sides of an inequality are multiplied, or divided, by the same negative number then the symbol is reversed.

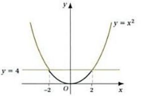
Quadratic inequalities

One example of a quadratic inequality is $x^2 > 4$. It has two sets of solutions. Consider $x^2 = 4$. This has two solutions; $x = 2$ and $x = -2$. Why? Because to solve the equation $x^2 = 4$ we take the square root of both sides of the equation.

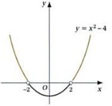
$$x^2 = 4 \;\Rightarrow\; x = \pm\sqrt{4} = \pm 2$$

So what do we know about x when $x^2 > 4$? We can see that $x > 2$ works. Check this result. What is the other solution? Is there a condition involving -2? We find that $x < -2$ is also a condition that works.

To see why the solutions are $x > 2$ and $x < -2$, think about the proposition $x^2 > 4$. Draw two graphs; $y = x^2$ and $y = 4$. Where is the parabola above the line $y = 4$? This is the same as asking for which values of x the graph of $y = x^2$ is above the line $y = 4$, or for which values $x^2 > 4$. The curve is above the line for $x > 2$ and for $x < -2$.



An alternative is to rearrange the original inequality. Then $x^2 > 4$ becomes $x^2 - 4 > 0$. This may not look simpler, but the new statement has a quadratic expression and a zero separated by an inequality symbol. A graph of $y = x^2 - 4$ can now be drawn, and we are looking for the points where $y > 0$ (that is, for points of the curve above the x-axis).



Since 2 and −2 are nc part of the solution, tl dots here are left unshaded.

Find the points where the curve crosses the x-axis (by solving $x^2 - 4 = 0$, or using your graphical calculator). Notice that the curve is above (greater than) the x-axis ($y = 0$) for $x > 2$ and $x < -2$.

Graphic calculat support pack

This technique of sketching the graph is a useful way of checking that no solutions have been lost.

Example 2

Solve:

a $x^2 - 7x < -10$ **b** $x^2 - 3x - 5 \geq 0$ **c** $x^2 + x + 1 \leq 0$

Solution

a This is a quadratic inequality. It can be rewritten as a quadratic expression and a zero separated by an inequality symbol.

$$x^2 - 7x < -10 \Rightarrow x^2 - 7x + 10 < 0$$

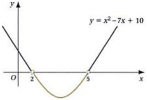
The quadratic expression can be factorised. Recall 'PAFF' from Chapter 1.

P: 10 A: −7 F: −2, −5

Recall that since the coefficient of x^2 is 1 the expression can be factorised from this step.

Then $x^2 - 7x + 10 < 0$ becomes $(x - 2)(x - 5) < 0$.
What does this expression suggest about x?
If the inequality symbol was an equality (that is, $(x - 2)(x - 5) = 0$) then $x = 2$ or $x = 5$ would be the solution. Since we have an inequality, these **critical values** should be examined more closely. Begin by sketching the curve $y = x^2 - 7x + 10$. Notice how it crosses the x-axis at $x = 2$ and $x = 5$.

Critical values are points where the quadratic expression changes sign.



The parabola is below the x-axis for all values of x between $x = 2$ and $x = 5$. This means $x^2 - 7x + 10$ is negative for all these values of x. So $x^2 - 7x + 10 < 0$ when $x > 2$ and $x < 5$.
Another way of writing this set of inequalities is as a 'sandwich': $2 < x < 5$.
Notice how the x appears between the values of 2 and 5 found in the factorisation process.

b Try the technique used in **a**, and see what happens.
Try factorising the quadratic expression $x^2 - 3x - 5$, using PAFF.

P: −5 A: −3 F: ?

Values for F cannot be found easily so this quadratic expression cannot be factorised using PAFF, but the critical values can be identified by solving $x^2 - 3x - 5 = 0$. Since PAFF isn't working, use the quadratic formula with $a = 1$, $b = -3$ and $c = -5$.

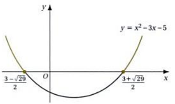
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

◀ **Remember to quote the formula.**

$$= \frac{3 \pm \sqrt{9 - 4 \times 1 \times (-5)}}{2}$$

$$= \frac{3 \pm \sqrt{9 + 20}}{2} = \frac{3 \pm \sqrt{29}}{2}$$

Notice that these values of x are both irrational due to the $\sqrt{29}$ term. Now sketch the curve of $y = x^2 - 3x - 5$. (Use a graphical calculator if you have one.)

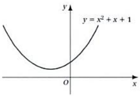


The critical values ar
part of the solution s
the dots on the graph
are shaded.

Notice that the curve is on or above the x-axis (that is $y \geq 0$) when $x \leq \frac{1}{2}(3 - \sqrt{29})$ and $x \geq \frac{1}{2}(3 + \sqrt{29})$. These inequalities are separate and cannot be condensed into a 'sandwich'.

This is because x mu
be smaller than the l
critical value and big
than the highest criti
value.

c Remember that the technique has been to identify critical values. By sketching the graph a set of inequalities has been identified where the graph is above or below the axis.
Sketch the graph of $y = x^2 + x + 1$. What happens?



The graph doesn't cross the axis. The expression has no critical values. Check this by trying to factorise $x^2 + x + 1$, or use the quadratic formula. Since the curve is always above the x-axis, $x^2 + x + 1$ is never negative, and $x^2 + x + 1 \leq 0$ has no solutions.

Alternatively we can
complete the square.
$x^2 + x + 1 \equiv (x + \frac{1}{2})^2$
$\geq \frac{3}{4}$
So there are no real
solutions.

In summary the technique for solving quadratic inequalities is as follows:

① Establish zero on one side of the inequality symbol. This means the quadratic expression can then be tested for being positive (>0) or negative (<0).

② Establish the critical values. These are the values of x that make the quadratic expression equal zero.

③ Identify the set of inequality solutions. Often this can be done by sketching a suitable graph.

④ Decide which sides of the critical points form the solution set.

An alternative method is to check the sign of the expression below the smallest critical value, between the critical values and above the largest critical value.

Example 3

Solve $2x^2 - 9x + 9 \geq 0$.

Solution

Notice that the first step has been done. This is a quadratic expression that needs to be positive (≥ 0). Now use PAFF to factorise the expression $2x^2 - 9x + 9$. ◀ ② **Identify the critical values.**

P: $2 \times 9 = 18$ A: -9 F: $-6, -3$

F: $2x^2 - 9x + 9 = 2x^2 - 6x - 3x + 9$

$$= 2x(x - 3) - 3(x - 3)$$

$$= (x - 3)(2x - 3)$$

So the critical values are 3, when $x - 3 = 0$, and $\frac{3}{2}$, when $2x - 3 = 0$.
Notice that the problem has changed from solving $2x^2 - 9x + 9 \geq 0$ to solving $(x - 3)(2x - 3) \geq 0$.
We will use the method where we check the sign of the expression against the critical values. Look at the sign of $(x - 3)(2x - 3)$ by comparing the signs of the separate factors.

	$x < \frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2} < x < 3$	3	$x > 3$
$(x-3)$	−		−		+
$(2x-3)$	−		+		+
$(x-3)(2x-3)$	$(-) \times (-)$ positive		$(-) \times (+)$ negative		$(+) \times (+)$ positive

Using a number line showing the critical values, check the sign of the quadratic for values on either side of each critical value.

So $(x - 3)(2x - 3)$ is positive when values of x are smaller than the least critical value, and larger than the highest critical value.
The set of solutions for $2x^2 - 9x + 9 \geq 0$ is $x \leq \frac{3}{2}$ and $x \geq 3$. Check this result by sketching the graph of $y = 2x^2 - 9x + 9$.

2.4 Inequalities
Exercise
Technique

1 Solve these linear inequalities:

a $x + 3 > 8$ **b** $2x + 3 \leq 7$

c $3x + 2 \geq x + 8$ **d** $\dfrac{2 + x}{3} < x + 4$

1 d

2 Solve these quadratic inequalities:

a $x^2 > 9$ **b** $x^2 \leq 25$ **c** $x^2 - 49 > 0$ **d** $x^2 - 64 < 0$

3 Solve these quadratic inequalities:

a $x^2 + 8x + 15 > 0$ **b** $x^2 + 5x - 6 \geq 0$ **c** $x^2 + 7x + 10 < 0$

d $x^2 - 2x - 15 < 0$ **e** $x^2 - 5x + 6 \leq 0$ **f** $x^2 + 3x - 4 > 0$

4 Transform these statements into quadratic inequalities involving zero. Solve the inequality in each case.

a $x^2 - 10x \geq 24$ **b** $x^2 + x > 6$ **c** $x^2 \leq 11x - 24$

d $x^2 + 6x > -9$ **e** $x^2 < 4x + 77$ **f** $x^2 + 4 \leq 4x$

5 Solve these quadratic inequalities:

a $3x^2 + 7x + 2 > 0$ **b** $7x^2 + 22x + 3 \leq 0$ **c** $3x^2 + 5x + 2 < 0$

d $3x^2 + 1 > 4x$ **e** $2x^2 \geq 5x + 3$ **f** $3x^2 + x > 2$

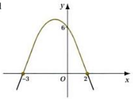
5 a

Contextual

1 Solve the inequality $x + 3 > x^2$, leaving your answer in the form $a + b\sqrt{c}$ where a, b and c are rational.

1

2 Find an inequality represented by the highlighted section of this graph.

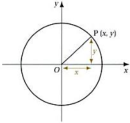


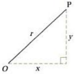
3 Find the set of values of x for which $(3x - 1)^2 < 3x^2 + 13$.

4 Solve the equation $x^2 - 5\sqrt{2}x + 12 = 0$, writing your answer using surds. Hence, or otherwise, solve $x^2 - 5\sqrt{2}x + 12 < 0$.

2.5 The Equation of a Circle

Consider a circle of radius r whose centre is at the origin, and let P (x, y) be any point on the circle. This means the distance OP must always be equal to the radius of the circle.





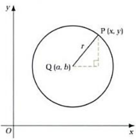
Pythagoras' theorem can be used to find the relationship between x, y and r.

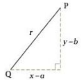
The equation of a circle of radius r whose centre is at the origin (0, 0) is:

$$x^2 + y^2 = r^2$$

Check that this is true for points on the circle where P has negative coordinates.

What happens if the centre of the circle is moved to a new position Q (a, b)? Again, let P (x, y) be some point on the circumference of the circle. Notice that PQ is a radius of length r. A new right-angled triangle can be drawn on PQ so that the shorter sides are parallel to the x-axis and y-axis respectively.





The lengths of these sides are $(x - a)$ and $(y - b)$. Check this from the diagram. Now use Pythagoras' theorem on this triangle. What happens?

$$(x - a)^2 + (y - b)^2 = r^2$$

This is the equation of a circle of radius r whose centre is the point (a, b).

Example 1

State the radius and centre of the following circles:

a $(x-7)^2+(y+2)^2=36$

b $(x+1)^2+(y-5)^2=23$

Solution

a Compare $(x-7)^2+(y+2)^2=36$ with $(x-a)^2+(y-b)^2=r^2$. The equation is in the same form with $a=7, b=-2$ and $r=6$.
So $(x-7)^2+(y+2)^2=36$ is the equation of a circle of radius 6, centre $(7,-2)$.

Note that since the radius is a distance it is always taken to be positive.

b $(x+1)^2+(y-5)^2=23$ can be compared with $(x-a)^2+(y-b)^2=r^2$, to give $a=-1, b=5$ and $r=\sqrt{23}$.
So $(x+1)^2+(y-5)^2=23$ is the equation of a circle of radius $\sqrt{23}$ whose centre is $(-1,5)$.

Example 2

a The point $(k,2)$ lies on the circle $x^2+y^2=13$. Find the values of k.

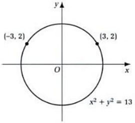
b The point $(k,0)$ lies on the circle with centre $(7,2)$ and radius $\sqrt{8}$. Find the possible values of k.

Solution

a Substitute $(k,2)$ into the equation of the circle.

$$\text{Then } x^2+y^2=13 \Rightarrow k^2+2^2=13$$
$$\Rightarrow k^2+4=13$$
$$\Rightarrow k^2=9$$
$$\Rightarrow k=\pm\sqrt{9}=\pm 3$$

So there are two possible values of k. These can be interpreted geometrically by sketching a diagram of the circle.



b The equation for a circle of radius $\sqrt{8}$, centre $(7, 2)$ is given by

$$(x-7)^2 + (y-2)^2 = 8$$

Now put $x = k$ when $y = 0$ into this equation.

$$(k-7)^2 + (0-2)^2 = 8$$
$$\Rightarrow \quad (k-7)^2 + 4 = 8$$
$$\Rightarrow \quad (k-7)^2 = 4$$
$$\Rightarrow \quad (k-7) = \pm\sqrt{4} = \pm 2$$
$$\Rightarrow \quad k = 7 \pm 2$$

So $k = 9$ or $k = 5$.

We have seen that the equation of a circle of radius r and centre (a, b) can be written $(x-a)^2 + (y-b)^2 = r^2$. It can also be written in another form by multiplying out the bracketed terms and then collecting like terms.

$$(x-a)^2 + (y-b)^2 = r^2$$
$$\Rightarrow \quad (x^2 - 2ax + a^2) + (y^2 - 2by + b^2) = r^2$$
$$\Rightarrow \quad x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - r^2) = 0$$

Since a, b and r are all constants this can be 'simplified' to the **general form** of the equation for a circle (by writing $a^2 + b^2 - r^2 = c$).

$$\mathbf{x^2 + y^2 - 2ax - 2by + c = 0}$$

Notice that in this form:

- The centre (a, b) can be identified from the coefficients of the x and y terms.
- The radius is not as straightforward to identify as it was in the other form.

Example 3

a Find the equation of a circle of radius $\sqrt{7}$ and centre $(3, -2)$ in its general form.

b The equation of a circle is $x^2 + y^2 - 2x + 4y - 4 = 0$. Find the centre and radius of the circle.

Solution

a If the centre is $(3, -2)$ and the radius $\sqrt{7}$ then the equation of the circle is

$$(x-3)^2 + (y+2)^2 = (\sqrt{7})^2$$

Expanding the brackets, $(x^2 - 6x + 9) + (y^2 + 4y + 4) = 7$

$$\Rightarrow \quad x^2 + y^2 - 6x + 4y + (9 + 4 - 7) = 0$$

$$\Rightarrow \quad x^2 + y^2 - 6x + 4y + 6 = 0$$

b Compare $x^2 + y^2 - 2x + 4y - 4 = 0$ to the general form of the equation for a circle, $x^2 + y^2 - 2ax - 2by + c = 0$. Notice that a and b can be found by equating coefficients of x and y.

Equating coefficients of x, $\quad -2 = -2a, \quad$ so $a = 1$.

Equating coefficients of y, $\quad 4 = -2b, \quad$ so $b = -2$.

So the centre of the circle, (a, b), is $(1, -2)$.
Now equating the constant terms in each equation, $c = -4$.
Recall that $c = a^2 + b^2 - r^2$

So $\quad a^2 + b^2 - r^2 = -4$

$\Rightarrow \quad (1)^2 + (-2)^2 - r^2 = -4$

$\Rightarrow \quad 1 + 4 - r^2 = -4$

$\Rightarrow \quad r^2 = 1 + 4 + 4$

$\Rightarrow \quad r^2 = 9$

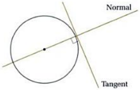
$\Rightarrow \quad r = 3$

We only take the positive square root because we are lookir at something we knov to be a positive value the radius.

So the centre of the circle is $(1, -2)$ and the radius is 3.

Tangents and normals

A **tangent** to a circle is a straight line that touches the circle at one distinct point. A **normal** is the straight line perpendicular to the tangent that passes through the point of contact between the tangent and the circle. Notice that the normal is an extension of a diameter.



Example 4

Find the equation of the tangent to the circle $x^2 + y^2 - 3x - y - 2 = 0$ at the point $(3, 2)$. Then find the equation of the normal.

Solution

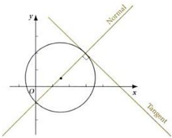
Compare $x^2 + y^2 - 3x - y - 2 = 0$ to the general form $x^2 + y^2 - 2ax - 2by + c = 0$. The values of a and b can be established by equating coefficients.

Both the tangent and the normal pass through (3, 2); we know one point on each line. The normal passes through the centre of the circle, so start by finding the centre, (a, b), of the circle.

Equating coefficients of x, $\quad -3 = -2a, \quad$ so $a = \frac{3}{2}$.

Equating coefficients of y, $\quad -1 = -2b, \quad$ so $b = \frac{1}{2}$.

So the centre of the circle, $(a, b) = (\frac{3}{2}, \frac{1}{2})$.



The gradient of the normal can now be found by using the points (3, 2) and $(\frac{3}{2}, \frac{1}{2})$.

$$\text{gradient of normal} = \frac{2 - \frac{1}{2}}{3 - \frac{3}{2}}$$

$$= \frac{\frac{3}{2}}{\frac{3}{2}} = 1$$

Since the tangent is perpendicular to the normal its gradient must be -1. We now know the gradient of the tangent and the coordinates of a point on the tangent (3, 2), so the equation of the tangent is given by

Recall that the product of the gradients of two perpendicular lines is always -1.

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow \quad y - 2 = -1(x - 3)$$

$$\Rightarrow \quad y - 2 = -x + 3$$

$$\Rightarrow y + x - 5 = 0$$

Similarly, we know the gradient of the normal and the coordinates of a point on the normal, so the equation of the normal is given by

We knew another point on the normal (the centre of the circle), so we could also have used the technique for finding the equation of a straight line knowing two points on that line.

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow \quad y - 2 = 1(x - 3)$$

$$\Rightarrow \quad y - 2 = x - 3$$

$$\Rightarrow y - x + 1 = 0$$

2.5 The Equation of a Circle
Exercise
Technique

1 Find the centre and radius of each of the following circles:

a $(x-5)^2+(y-3)^2=7^2$ b $(x+6)^2+(y+1)^2=25$ 1 a

c $x^2+(y+7)^2=121$ d $(x+1)^2+(y-2)^2=11$

2 Write the equations of the circles with the given centres and radii in the form $x^2+y^2-2ax-2by+c=0$:

a centre $(-1,3)$, radius 2 b centre $(2,-1)$, radius 3

c centre $(-2,0)$, radius $\sqrt{5}$ d centre $(3,-3)$, radius $3\sqrt{2}$

3 Find the centre and radius of the following circles.

a $x^2+y^2-6x-2y+6=0$ b $x^2+y^2+2x-4y+1=0$ 3 b

c $x^2+y^2+2x+8y+8=0$ d $x^2+y^2+4y+1=0$

Contextual

1 The point $(3,k)$ lies on the circle $x^2+y^2-6x-4y-51=0$. Find the values of k.

2 The point $(k,0)$ lies on the circle with centre $(4,1)$ and radius $\sqrt{10}$. Find two possible values of k.

3 The point A lies on the circle with centre $(1,3)$ and radius $\sqrt{5}$. Given that A lies on the y-axis, find the possible coordinates of A.

4 Find the equation of the tangent to the circle $x^2+4x+y^2=21$ at the point $(1,4)$. Find also the equation of the normal at this point.

5 A straight line touches the circle $x^2-4x+y^2-10y-71=0$ at the point $(8,-3)$. Find the equation of the line and the equation of any line perpendicular to it passing through the point $(4,3)$.

6 The points A $(-7,7)$ and B $(1,1)$ form the diameter of a circle. Find the equation of the circle.

7 Find the length of the tangent from the point $(9,8)$ to the circle $x^2+y^2-2x-4y=31$.

Consolidation

Exercise A

1 A, B, C are the points with coordinates $(4, 7)$, $(-1, 2)$ and $(6, 1)$ respectively.

a Prove that the triangle ABC is isosceles. State the coordinates of the mid-point, M, of AC and find the area of the triangle.
b Find the equation of the line BM.
c Find the equation of the line through A, perpendicular to BC.
d Find the coordinate of the point H where these two lines meet, and deduce that CH is perpendicular to AB.

(OCSEB)

2 The coordinates of the points A and B are $(2, 3)$ and $(4, -3)$ respectively. Find the length of AB and the coordinates of the mid-point of AB.

(UCLES)

3 P, Q, R are the points whose coordinates are $(2, 4)$, $(8, -4)$ and $(14, 8)$ respectively.

a Find the equations of the perpendicular bisectors of the lines PQ and PR.
b If the two bisectors meet at C, calculate the coordinates of C and show that $CP^2 = 50$.
c Deduce the equation of the circle through P, Q, and R in the form $x^2 + y^2 + px + qy + r = 0$.

(OCSEB)

4 A line through the origin with gradient m cuts the fixed circle in Fig. 1 in two points provided that $2(2m+1)^2 > 9(m^2+1)$. Show that this inequality is equivalent to $m^2 - 8m + 7 < 0$ and find the solution set for m.

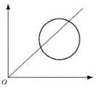


Fig. 1

(OCSEB)

5 Find the length of the tangent from the point (6, 2) to the circle $x^2 + y^2 + 2x = 9$.

6 Solve the inequality $(x - 3)(x - 6) > x + 9$.

(*AEB*)

Exercise B

1 The coordinates of the points A and B are (3, 2) and (4, −5) respectively. Find the coordinates of the mid-point of AB, and the gradient of AB. Hence find the equation of the perpendicular bisector of AB, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(*UCLES*)

2 The straight line P passes through the point (10, 1) and is perpendicular to the line R with equation $2x + y = 1$. Find the equation of P. Find also the coordinates of the point of intersection of P and R and deduce the perpendicular distance from the point (10, 1) to the line R.

(*UCLES*)

3 Find the length of the tangent from the point (7, 6) to the circle $x^2 + y^2 - 2x = 15$.

4 Find the equation of the straight line that is parallel to $y + 20x = 90$ and passes through the point (4, −10).

(*NEAB*)

5 Find the set of values for which $\dfrac{x}{x+4} > 2$.

(*ULEAC*)

Hint: Solve $x > 2(x +$ and $x + 4 > 0$.

6 A circle, centre P, passes through A (1, 1), B (−2, 2) and C (−7, −3).

a Find the equation of the perpendicular bisector of AB.
b Find the equation of the perpendicular bisector of BC.
c Using your answers to **a** and **b**, solve the equations simultaneously to find centre P.
d Find the distance *AP*.
e Hence write down the equation of the circle.

Applications and Activities

Constructing a circle through three known points

1 Mark three points anywhere on a piece of paper. Now try to construct a circle that passes through all three. Can it always be done?

2 Mark the three points on a piece of graph paper so that their coordinates can be read. Repeat the problem but this time find:

- **a** the coordinates of the centre of the circle
- **b** the radius of the circle and
- **c** the equation of the circle.

Summary

- The formula for the distance between points A (x_1, y_1) and B (x_2, y_2) is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- The **gradient** of a line is a measure of its steepness.
- gradient $= \dfrac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$
- **Parallel** straight lines have the same gradient.
- The product of the gradients of **perpendicular** straight lines is -1.
- The equation of a straight line is generally written in the forms $y = mx + c$ $(y = a_1x + a_0)$ and $ax + by + c = 0$.
- The equation of a straight line with gradient m passing through (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

- The equation of a straight line passing through (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

- The mid-point M of a line joining the points A (x_1, y_1) and B (x_2, y_2) has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- The symbols $>, \leq, <, \geq$ mean 'greater than', 'less than or equal to', 'less than' and 'greater than or equal to', respectively.

- Linear equalities can be represented on a number line using $\circ\!\!\rightarrow$ and $\bullet\!\!\rightarrow$ as appropriate.
- Quadratic inequalities can be solved by rearranging them as factorised quadratic expressions with zero on one side. Solutions can be checked by drawing graphs.
- The equation of a circle has general forms

  $$(x-a)^2+(y-b)^2=r^2$$

  and

  $$x^2+y^2-2ax-2by+c=0$$

- A circle, $(x-a)^2+(y-b)^2=r^2$, has centre (a,b) and radius r.
- The **normal** to a circle at a given point is perpendicular to the tangent at that point, and passes through the centre of the circle.

3 Trigonometry I

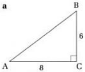
What you need to know

- How to use Pythagoras' theorem.
- Factorisation methods, including factorisation of quadratic equations.
- How to write down the sine, cosine and tangent ratios for acute angles.
- How to find the area of a triangle.
- That the term solving a triangle means finding the lengths of the unknown sides and the sizes of the unknown angles.
- How to calculate bearings.

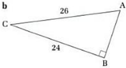
Review

1 Use Pythagoras' theorem to find the length of side AB in the following triangles:

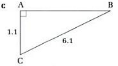
a



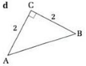
b



c



d



Pythagoras of Samos (c. 560–c. 480BC)
Pythagoras provided the first logical proof of the theorem $a^2 + b^2 = c^2$ for right-angled triangles, and used letters on geometric figures.

2 Factorise each of the following expressions and solve the equations:

a $a^2 - b^2$

b $c^2 + 3c + 2$

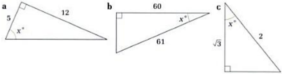
c $h^2 + h - 12$

d $k^2 - 7k + 12$

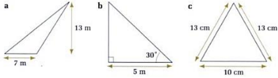
e $p^2 - 8p + 12 = 0$

f $3p^2 + 14p - 5 = 0$

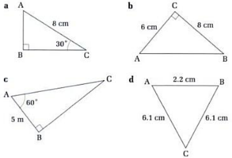
3 Write down the values of $\sin x$, $\cos x$ and $\tan x$ for the following triangles:



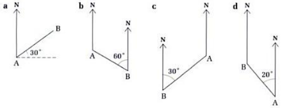
4 Calculate, to three significant figures, the areas of the following triangles:



5 Solve the following triangles:

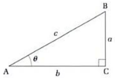


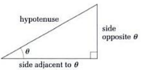
6 For each of the following diagrams write down the bearing of B from A:



3.1 Trigonometric Functions

Trigonometry is the study of angle measurement, and in particular the study of triangle measurement and calculation. In order to distinguish between angles and lengths of sides the convention of capital letters for vertices and lower case letters for the corresponding opposite side is adopted.





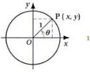
Right-angled triangles are used to define the three basic trigonometric functions for some acute angle θ; sine, cosine and tangent.

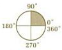
$$\sin\theta = \frac{a}{c} = \frac{\text{side opposite } \theta}{\text{hypotenuse}} \qquad \cos\theta = \frac{b}{c} = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}}$$

$$\tan\theta = \frac{a}{b} = \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta}$$

This principle can be used to define the sine, cosine and tangent of any angle θ.

Draw perpendicular axes Ox and Oy, and a circle centred on the origin, with radius 1 unit. Then θ will fix some point P on the circle.





The first quadrant

The coordinates of P (x, y) are then $(\cos\theta, \sin\theta)$. Now adopt the convention that θ is measured anti-clockwise from the positive x-axis. The **quadrant** between the positive x-axis and the positive y-axis is called the **first quadrant**. In this quadrant θ is always acute.

The **second quadrant** is between the positive y-axis and the negative x-axis. In this quadrant θ is always obtuse. When θ is obtuse (greater than

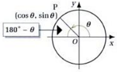
The origins of the development of trigonometry are obscure. It was largely a Hindu invention, a mixture of algebra and geometry, used whenever lengths and angles needed to be calculated.

The abbreviations sin and cos were introduced by William Oughtred in his book published in 1642. He also introduced the × sign for multiply.

90°) $\sin\theta$ and $\cos\theta$ are equal in magnitude to the sine and cosine ratio of the acute angle $(180° - \theta)$. So in the second quadrant the coordinates of P are still $(\cos\theta, \sin\theta)$, but note that in this quadrant $\sin\theta$ is positive, and $\cos\theta$ is negative. ◄ $\sin(180° - \theta) \equiv \sin\theta$

$\cos(180° - \theta) \equiv -\cos\theta$

The magnitude is the numerical value, or si of the trigonometric ratio, ignoring the sig (positive or negative).



The second quadrant

Example 1

Find $\cos 147°$ as a trigonometric ratio of an acute angle.

Solution

We know that 147° lies in the second quadrant and, for obtuse θ, $\cos\theta = -\cos(180° - \theta)$.

So $\cos 147° = -\cos(180° - 147°) = -\cos 33°$.

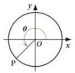
Check this result on a calculator.

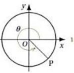
By making θ a reflex angle, we can extend these results into the third and fourth quadrants. In the third quadrant both $\sin\theta$ and $\cos\theta$ are negative. In the fourth quadrant $\cos\theta$ is positive and $\sin\theta$ is negative. (Think carefully about the coordinates of the point P).

$\sin(\theta - 180°) = -\sin$
$\cos(\theta - 180°) = -\cos$

$\sin(360° - \theta) = -\sin$
$\cos(360° - \theta) = \cos\theta$

The third quadrant





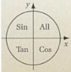
The fourth quadrant

Remembering the definition of $\cos\theta$ and $\sin\theta$ as the coordinates of P, the gradient of the line OP gives us $\tan\theta$.

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

This allows us to establish the quadrants in which each of the three trigonometric ratios are positive. One way of remembering which trigonometric ratios are positive and which negative in each quadrant is to remember only the positive ones. Think about the coordinates of P $(\cos\theta, \sin\theta)$ and remember that $\tan\theta$ is positive when $\sin\theta$ and $\cos\theta$ have the same sign.

All the trigonometric ratios are positive in the first quadrant, sine is positive in the second, tangent is positive in the third and cosine is positive in the fourth quadrant. There are several good mnemonics (aids to memory), such as **All Silly Tom Cats**; **All Squirrels Take Chestnuts**; **All Silver Tea Cups**.



Example 2

Write $\tan 227°$ as a trigonometric ratio of an acute angle.

Solution

We know that $227°$ lies in the third quadrant, and that $\tan\theta$ is positive in the third quadrant. So $\tan 227° = \tan(227° - 180°) = \tan 47°$

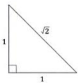
Check this result on a calculator.

Special angles

Some acute angles are special because they occur so frequently. Two triangles in particular are very useful for finding the trigonometric ratios of these angles. These triangles have the advantage of giving exact results and not decimal approximations.

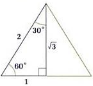
- An isosceles right-angled triangle with sides 1 unit.

From Pythagoras' theorem, $1^2 + 1^2 = 2$.



- Half of an equilateral triangle of side 2 units.

From Pythagoras' theorem, $2^2 - 1^2 = 3$.



Using these triangles we have the following special results.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Copy and memorise them.

This now allows the ratios of related angles in the second, third and fourth quadrants to be evaluated exactly.

Example 3

a Find $\sin 150°$.

b Find $\cos 330°$.

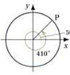
c Find $\sin 410°$ as a trigonometric ratio of an acute angle.

Solution

a We know that 150° lies in the second quadrant, where sine is positive. So $\sin 150° = \sin(180° - 150°) = \sin 30° = \frac{1}{2}$.

b We know that 330° is in the fourth quadrant, and cosine is positive in the fourth quadrant. So $\cos 330° = \cos(360° - 330°) = \cos 30° = \frac{\sqrt{3}}{2}$.

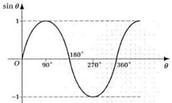
c Angles outside the range 0°–360° always lie in one of the four quadrants. We find that $410° - 360° = 50°$, so 410° is in the first quadrant, where sine is positive.
So $\sin 410° = \sin(410° - 360°)$
$= \sin 50°$



Check this result on a calculator.
Recall that $\frac{\sqrt{3}}{2}$ is irrational. If you chec this result on a calculator your scree will probably show 0.8660...

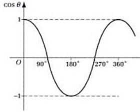
Graphs of the trigonometric functions

Now draw a new pair of axes and plot the angle θ along the x-axis and the y-coordinate of P ($\sin\theta$) along the y-axis. This gives a continuous curve. This curve, or wave, repeats itself every 360°, so the sine curve is said to have a **period** of 360°. The curve has a maximum value of 1 (when $\theta = 90°$) and a minimum of -1 (when $\theta = 270°$). These correspond to P being at the top and bottom of the circle.



The systematic stu trigonometry is attributed to Hipparc an Alexandrian astron who lived around 150 He made a table of si and used it to find th distance between the Moon and the Earth.

You can draw a similar graph plotting the angle θ along the x-axis and the x-coordinate of P ($\cos\theta$) along the y-axis. The graph of $\cos\theta$ is the same shape as that of $\sin\theta$, but it has been shifted by 90°. The 90° is sometimes referred to as the **phase difference** between the two graphs.



The similarities can be described using the following equations.

$\sin(\theta + 90^\circ) = \cos\theta$ $\quad$ $\sin(90^\circ - \theta) = \cos\theta$

$\cos(\theta + 90^\circ) = -\sin\theta$ $\quad$ $\cos(90^\circ - \theta) = \sin\theta$

Example 4

Show that $\sin 120^\circ = \cos 30^\circ$.

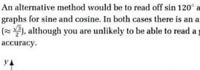
Solution

$$\begin{aligned}\sin 120^\circ &= \sin(180^\circ - 120^\circ)\\ &= \sin 60^\circ\\ &= \tfrac{\sqrt{3}}{2}\\ &= \cos 30^\circ\end{aligned}$$

120° is in the second quadrant, so sin is positive.

Recall that $\frac{\sqrt{3}}{2}$ is the trigonometric ratio of a special angle.

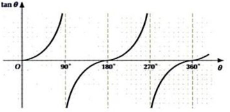
An alternative method would be to read off $\sin 120^\circ$ and $\cos 30^\circ$ from the graphs for sine and cosine. In both cases there is an answer of 0.8660 . . . ($\approx \frac{\sqrt{3}}{2}$), although you are unlikely to be able to read a graph to this level of accuracy.



The graph of $\tan\theta$ doesn't look like either of the other two graphs. This is because $\tan\theta$ is defined as

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

The denominator, $\cos\theta$, has the value 0 when $\theta = 90°$. This means that the graph of $\tan\theta$ will not be continuous. To show this on the graph we use dotted lines called **asymptotes**. Like $\sin\theta$ and $\cos\theta$, $\tan\theta$ is periodic, but this time the period is 180° instead of 360°.



You should memorise the main features of the sine, cosine and tangent graphs. The main features are the general shape, maximum and minimum points, intersections with axes and positions of asymptotes.

Trigonometric equations

A trigonometric equation is one containing a trigonometric function such as sine, cosine or tangent. Solving these equations means finding values of the angle that satisfy the equation. Usually the range of angles that are acceptable as solutions will be restricted. The restriction could be due to the nature of the problem, or imposed by the person setting the question. To find angles within these ranges the trigonometric graphs can be used.

In addition to finding the sine, cosine and tangent of a known angle, a calculator can be used to find an angle with a particular sine, cosine or tangent. The inverse trigonometric functions, written $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, are used. On a calculator they are usually located above the 'sin', 'cos' and 'tan' function keys. Sometimes an 'inverse', 'shift', '2nd function' or 'arc' key needs to be pressed first.

Alternative notation [for] inverse trigonometric [functions/]angles that you may come across is **arcsin**, **arccos** and **arctan** (a[nd] sometimes arsin, arc[os] and artan).

If a calculator is used to find an angle it will give an answer called the **principal value**. Other solutions can then sometimes be found by adding multiples of the period for that function. Alternatively, once the principal value is known, the symmetry of the trigonometric graphs can be used to find solutions.

Example 5

a Find values of θ for which $4\sin\theta = 3$ such that $0° \leq \theta \leq 360°$.

b Solve the equation $\tan\theta = -2$ for $0° \leq \theta \leq 360°$.

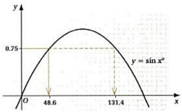
Solution

a $4\sin\theta = 3 \Rightarrow \sin\theta = \frac{3}{4}$ ◀ Recall that $\Rightarrow$ means 'implies'.

$\theta = \sin^{-1}(0.75) = 48.6°$ (3 s.f.)

First isolate the trigonometric ratio (sine), and notice that $\sin\theta$ is positive, so we expect solutions in the first and second quadrants.

The solution in the first quadrant is 48.6°.
The solution in the second quadrant is $180° - 48.6° = 131.4°$

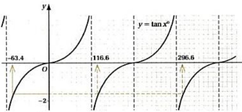


b $\tan\theta = -2 \Rightarrow \theta = \tan^{-1}(-2)$

$\theta = -63.4°$ (3 s.f.)

In this case the calculator gives a principal value outside the acceptable range, and negative.

Add multiples of 180°, the period of $\tan\theta$, to find solutions that lie within the acceptable range.



$\theta = -63.4° + 180° = 116.6°$

$\theta = -63.4° + 360° = 296.6°$

$\theta = -63.4° + 540° = 476.6°$, which is outside the acceptable range.

There are two solutions; $\theta = 116.6°$ and 296.6°. (Notice that they are in the second and fourth quadrants, where $\tan\theta$ is negative.)

Graphical calculator support pack

The reciprocal ratios

There are three other trigonometric ratios, which are known as the **reciprocal ratios**. These are cosecant (cosec), secant (sec) and cotangent (cot).

These ratios are used less frequently but ar applied in astronomy navigation and mechanics, particula in projectile motion.

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Example 6

Find:

a $\operatorname{cosec} 30^\circ$ **b** $\cot 60^\circ$ **c** $\sec 42^\circ$

Solution

a $\operatorname{cosec} 30^\circ = \dfrac{1}{\sin 30^\circ}$

$= \dfrac{1}{\frac{1}{2}} = 2$

Recall that 30° is a special angle.

b $\cot 60^\circ = \dfrac{1}{\tan 60^\circ} = \dfrac{1}{\sqrt{3}}$

c $\sec 42^\circ = \dfrac{1}{\cos 42^\circ}$

$= \dfrac{1}{0.7431} = 1.3457$ (4 d.p.)

Work to four decimal places with the trigonometric ratios.

Example 6c demonstrates the need to work to an appropriate degree of accuracy when using trigonometric functions on a calculator.

If you use full screen accuracy with cos 42 then $\sec 42^\circ = 1.3456$ (4 d.p.). Note the rounding error.

Example 7

Find one solution to $\operatorname{cosec}\theta = 3$.

Solution

$\operatorname{cosec}\theta = 3 \Rightarrow \dfrac{1}{\sin\theta} = 3$

$\Rightarrow \quad 1 = 3\sin\theta$

$\Rightarrow \sin\theta = \frac{1}{3}$

$\theta = \sin^{-1}(\frac{1}{3}) = 19.5^\circ$ (3 s.f.)

Notice that this solut is the principal value

From these examples we see that a useful strategy for solving trigonometric equations is as follows.

Step ① Rearrange the equation to make sine, cosine or tangent the subject.
Step ② Use a calculator (if necessary) to find the principal value.
Step ③ Using a graph or by adding multiples of the period find solutions in the acceptable range of angles.

3.1 Trigonometric Functions
Exercise
Technique

1 Write each of the following as trigonometric ratios of positive acute angles:

a	$\sin 120°$	**e**	$\tan 400°$
b	$\cos 165°$	**f**	$\cos(-137°)$
c	$\tan 220°$	**g**	$\sin(-29°)$
d	$\cos 305°$	**h**	$\sin(-697°)$

2 Write down the exact value of the following, leaving answers in terms of surds if appropriate:

a	$\cos 150°$	**e**	$\cos 120°$
b	$\sin 225°$	**f**	$\tan 420°$
c	$\tan 300°$	**g**	$\cos(-300°)$
d	$\sin 330°$	**h**	$\sin(-420°)$

2 a, c

3 Solve the following trigonometric equations for $0 \leq \theta \leq 360°$. Give your answers correct to one decimal place:

a	$\sin\theta = 0.314$	**d**	$3\tan\theta = \sqrt{2}$
b	$\cos\theta = -0.52$	**e**	$\sin\theta = \cos\theta$
c	$\tan\theta = 2.561$	**f**	$2\sin\theta = 3\cos\theta$

4 Find, correct to four significant figures, the value of:

a	$\operatorname{cosec} 39°$	**e**	$\cot 200°$
b	$\sec 41°$	**f**	$\operatorname{cosec} 307°$
c	$\cot 93°$	**g**	$\cot 420°$
d	$\sec 129°$	**h**	$\operatorname{cosec}(-15°)$

5 Find the principal value solutions to the following:

a	$5\sin\theta = -3$	**d**	$2\cot\theta - 3 = 0$
b	$\tan\theta + 3 = -7$	**e**	$4 - 3\tan\theta = 11$
c	$\sec\theta = 4$	**f**	$2\operatorname{cosec}\theta = 3$

Remember to work to 1 d.p. when finding angles.

5 f

6 Construct a table giving values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for appropriate values of θ in the range $-90° \leq \theta \leq 450°$. On separate pieces of paper, draw the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for this range of angles. Use the graphs to solve the equations:

a $\sin\theta = -0.5$
b $\cos\theta = 0.8$
c $\tan\theta = 3$

7 Draw the graph of $y = \sin\theta$ for values of θ in the range $-90^\circ \leq \theta \leq 90^\circ$. Use your graph to find:

a $\sin^{-1}(-0.5)$ **b** $\sin^{-1}(0.71)$

Contextual

1 The depth of water in a harbour, y metres, can be modelled by the equation $y = 5\sin(30t)^\circ + 12$, where t is the time in hours from midnight on a particular day.

a Draw the graph of this function over a period of 24 hours.
b From the graph, find the times of high and low tides.
c Use the graph to find the length of time for which the depth of water in the harbour is greater than 15 metres.

2 The height of a tide can be modelled by a function of the form $h = a\cos bt^\circ + c$, where h is the height in metres of the water and t is the time in hours after midnight. Find the values of a, b and c for the following tide table.

Tide	Time	Height (m)
High	00:00	12
Low	06:00	2
High	12:00	12
Low	18:00	2

3 The hours of daylight over a period of time can be modelled using a trigonometric equation. If n is the number of hours of daylight and x is the number of days from 1 January, then, $n = 12 - 6\cos(x + 10)^\circ$.

a Calculate the length of the daylight on 1 April, which is day 90.
b Use the equation to find the dates of the longest and shortest days.
c Comment on the reliability of the model.
d Suggest an amendment to the model.

3.2 Equations and Identities

Trigonometric equations can increase in complexity and solving them requires combining algebraic techniques and knowledge of the trigonometric functions.

Example 1

Solve $4\sin\theta - 3\cos\theta = 0$ for $0^\circ \le \theta \le 360^\circ$.

Solution

$$4\sin\theta - 3\cos\theta = 0 \Rightarrow 4\sin\theta = 3\cos\theta$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{3}{4}$$

$$\Rightarrow \tan\theta = 0.75$$

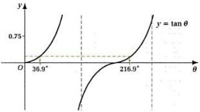
$$\theta = \tan^{-1}(0.75) = 36.9^\circ$$

Rearrange the equation using algebra.

Recall the definition of $\tan\theta$.

Remember that the calculator gives the principal value.

Tangent is positive in the first and third quadrants. The solution in the first quadrant is 36.9°, and the solution in the third quadrant is 216.9°. ◀ $180^\circ + 36.9^\circ = 216.9^\circ$.



Example 2

Solve $\cos\theta\sin\theta = 2\sin\theta$, for $0^\circ \le \theta \le 360^\circ$.

Solution

$$\cos\theta\sin\theta = 2\sin\theta \Rightarrow \quad \cos\theta\sin\theta - 2\sin\theta = 0$$

$$\Rightarrow \quad \sin\theta(\cos\theta - 2) = 0$$ ◀ Factorise.

$$\Rightarrow \sin\theta = 0 \text{ or } (\cos\theta - 2) = 0$$

When rearranging, remember that you can't divide throughout by $\sin\theta$, because $\sin\theta$ may be zero.

When $\sin\theta = 0$, $\theta = 0^\circ, 180^\circ, 360^\circ$.
When $\cos\theta = 2$ there are no solutions.
So the solutions are $\theta = 0^\circ, 180^\circ, 360^\circ$.

Example 3

Solve the equation $2\sin^2\theta + \sin\theta - 1 = 0$, for $0^\circ \leq \theta \leq 360^\circ$.

The notation $\sin^2\theta$ means $(\sin\theta)^2$.

Solution

The equation $2\sin^2\theta + \sin\theta - 1 = 0$ is a quadratic in $\sin\theta$. It can be solved directly, or by making a substitution for $\sin\theta$ (which can make the process look less complex).

- *Direct solution*

  $$2\sin^2\theta + \sin\theta - 1 = 0 \Rightarrow (2\sin\theta - 1)(\sin\theta + 1) = 0$$
  $$\Rightarrow 2\sin\theta - 1 = 0 \text{ or } \sin\theta + 1 = 0$$

  So $\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$.

- *Substitution*

  Let $y = \sin\theta$.

  $$2\sin^2\theta + \sin\theta - 1 = 0 \Rightarrow 2y^2 + y - 1 = 0$$
  $$\Rightarrow (2y-1)(y+1) = 0$$
  $$\Rightarrow 2y - 1 = 0 \text{ or } y + 1 = 0$$
  $$\Rightarrow y = \tfrac{1}{2} \text{ or } y = -1$$

  But $y = \sin\theta$, so $\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$.

Notice that this quadr equation in $\sin\theta$ has t distinct solutions for t given range of angles. assume that all quadra will always give two solutions.

Both methods give the same result. What else do you notice? You should notice that these are the trigonometric ratios for 'special angles'. When $\sin\theta = \frac{1}{2}$, $\theta = 30^\circ$, 150° for $0^\circ \leq \theta \leq 360^\circ$, and when $\sin\theta = -1$, $\theta = 270^\circ$. So the solutions are $\theta = 30^\circ, 150^\circ, 270^\circ$.

Another type of trigonometric equation is one involving brackets.

Example 4

Solve the equation $\cos(\theta + 30^\circ) = \frac{1}{2}$ for $-180^\circ \leq \theta \leq 180^\circ$.

Remember that this h many solutions.

Solution

$$\cos(\theta + 30^\circ) = \tfrac{1}{2} \Rightarrow (\theta + 30^\circ) = \cos^{-1}(\tfrac{1}{2})$$
$$\Rightarrow \theta + 30^\circ = \ldots, -60^\circ, 60^\circ, \ldots$$
$$\Rightarrow \theta = -90^\circ, 30^\circ \text{ in the given range}$$

So the solutions are $\theta = -90^\circ, 30^\circ$.

Check that these are solutions by substituting them back into the original equation.

Give solutions in the required range only.

$$\cos(-90^\circ + 30^\circ) = \cos(-60^\circ) = \tfrac{1}{2}$$
$$\cos(30^\circ + 30^\circ) = \cos 60^\circ = \tfrac{1}{2}$$

Example 5

Solve $\tan 2\theta = 1$, for $0^\circ \le \theta \le 360^\circ$.

Solution

If $\tan 2\theta = 1$ then 2θ must be a special angle. Note also the range of values for θ. The equation is in 2θ, so we must solve it for 2θ in the range $0^\circ \le 2\theta \le 720^\circ$.

$$\tan 2\theta = 1 \Rightarrow 2\theta = \tan^{-1}(1)$$
$$2\theta = 45^\circ, 225^\circ, 405^\circ, 585^\circ$$
$$\theta = 22.5^\circ, 112.5^\circ, 202.5^\circ, 292.5^\circ$$

Find all values of 2θ before dividing by 2.

Pythagorean identities

When equations involve more than one trigonometric function, and factorisation does not appear to be an obvious method, other techniques can be used. Often it is possible to substitute an equivalent expression for one already in the equation. Three results in particular are useful for this. They are known as **Pythagorean identities.** An **identity** is an equation that is true for all values of the variable. It is sometimes distinguished by the symbol $\equiv$ instead of $=$.

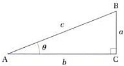
The Pythagorean identities are:

$\sin^2\theta + \cos^2\theta \equiv 1$ Identity 1

$\tan^2\theta + 1 \equiv \sec^2\theta$ Identity 2 ◀ $\sec^2\theta - \tan^2\theta \equiv 1$

$1 + \cot^2\theta \equiv \mathrm{cosec}^2\theta$ Identity 3 ◀ $\mathrm{cosec}^2\theta - \cot^2\theta \equiv 1$

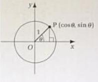
Identity 1 can be demonstrated by using Pythagoras' theorem.



$$\sin\theta = \frac{a}{c}$$
$$\cos\theta = \frac{b}{c}$$

$$\sin^2\theta + \cos^2\theta = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$$
$$= \frac{a^2+b^2}{c^2}$$
$$= \frac{c^2}{c^2} = 1$$

Alternatively we can use the unit circle $x^2 + y^2 = 1$. Now $x = \cos\theta$ and $y = \sin\theta$.



$\sin^2\theta + \cos^2\theta = 1$ works in all four quadrants.

The following examples illustrate how these results can be used to simplify trigonometric equations and establish further identities.

Example 6

Solve the equations:

a $2\sin^2\theta + 5\cos\theta + 1 = 0$ for $-180^\circ \le \theta \le 180^\circ$

b $3\cot^2\theta + 5\operatorname{cosec}\theta + 1 = 0$ for $0^\circ \le \theta \le 360^\circ$.

Solution

a $2\sin^2\theta + 5\cos\theta + 1 = 0 \Rightarrow 2(1-\cos^2\theta) + 5\cos\theta + 1 = 0$

Using identity 1.

$\Rightarrow \quad 2 - 2\cos^2\theta + 5\cos\theta + 1 = 0$

$\Rightarrow \quad 2\cos^2\theta - 5\cos\theta - 3 = 0$

Multiply out the bracket, collect like terms and make the highest power term positive.

This is a quadratic in $\cos\theta$ that can be factorised.

$(2\cos\theta + 1)(\cos\theta - 3) = 0$

So $\quad 2\cos\theta + 1 = 0$ or $\cos\theta - 3 = 0$

$\cos\theta = -\frac{1}{2}$ or $\cos\theta = 3$

$\cos\theta = -\frac{1}{2} \Rightarrow \theta = -120^\circ, 120^\circ$

Solutions in the second and third quadrants, where $\cos\theta$ is negative.

$\cos\theta = 3$ has no solutions

So the solutions are $\theta = \pm 120^\circ$.

b $3\cot^2\theta + 5\operatorname{cosec}\theta + 1 = 0 \Rightarrow 3(\operatorname{cosec}^2\theta - 1) + 5\operatorname{cosec}\theta + 1 = 0$

Using identity 3.

$\Rightarrow \quad 3\operatorname{cosec}^2\theta - 3 + 5\operatorname{cosec}\theta + 1 = 0$

$\Rightarrow \quad 3\operatorname{cosec}^2\theta + 5\operatorname{cosec}\theta - 2 = 0$

Factorising.

$\Rightarrow \quad (3\operatorname{cosec}\theta - 1)(\operatorname{cosec}\theta + 2) = 0$

$\Rightarrow \quad 3\operatorname{cosec}\theta - 1 = 0$ or $\operatorname{cosec}\theta + 2 = 0$

So $\operatorname{cosec}\theta = \frac{1}{3}$ or $\operatorname{cosec}\theta = -2$.

So now $\sin\theta = 3$ or $\sin\theta = -\frac{1}{2}$.

Using the definition of cosec θ.

$\sin\theta = 3$ has no solutions and $\sin\theta = -\frac{1}{2} \Rightarrow \theta = 210^\circ, 330^\circ$

So the solutions are $\theta = 210^\circ, 330^\circ$.

Example 7

Prove the identity $\sin\theta\tan\theta \equiv \sec\theta - \cos\theta$.

Solution

$$\begin{aligned}\sin\theta\tan\theta &= \sin\theta\frac{\sin\theta}{\cos\theta}\\ &= \frac{\sin^2\theta}{\cos\theta}\\ &= \frac{1-\cos^2\theta}{\cos\theta}\\ &= \frac{1}{\cos\theta}-\frac{\cos^2\theta}{\cos\theta}\\ &= \sec\theta-\cos\theta\end{aligned}$$

Using the definition of $\tan\theta$.

◀ **Using identity 1.**

Remembering the definition of $\sec\theta$.

That is, $\sin\theta\tan\theta \equiv \sec\theta - \cos\theta$.

Example 7 illustrates a useful procedure to adopt when establishing identities. Begin with the left-hand side (LHS) and then perform logical manipulations line by line until the form of the right-hand side (RHS) appears. At first this may appear very difficult, but keep in mind where you are trying to go. The range of equivalent forms mean that tasks like these are difficult for computers and symbol manipulators to do quickly.

Example 8

Prove the identity $\tan\theta + \cot\theta \equiv \sec\theta\operatorname{cosec}\theta$.

Solution

Beginning with the LHS,

$$\begin{aligned}\tan\theta+\cot\theta &\equiv \frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\\ &\equiv \frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\\ &\equiv \frac{1}{\sin\theta\cos\theta}\\ &\equiv \frac{1}{\sin\theta}\times\frac{1}{\cos\theta}\\ &\equiv \sec\theta\times\operatorname{cosec}\theta\end{aligned}$$

Writing the LHS in terms of $\sin\theta$ and $\cos\theta$.

◀ **Using identity 1.**

Recalling how fractions are multiplied.

And so we arrive at the RHS.

That is, $\tan\theta + \cot\theta \equiv \sec\theta\operatorname{cosec}\theta$.

Another useful technique that can help in proving identities is the ability to multiply by 1, with 1 written in a convenient form that allows further algebraic manipulation.

For example, $\dfrac{\sin\theta}{\sin\theta}\equiv 1$, $\dfrac{1+\cos\theta}{1+\cos\theta}\equiv 1$, $\dfrac{1-\tan\theta}{1-\tan\theta}\equiv 1$, and so on.

Example 9

Prove the identity $\dfrac{\cos\theta}{1-\sin\theta} \equiv \sec\theta + \tan\theta$.

Solution

Beginning with the LHS, $\dfrac{\cos\theta}{1-\sin\theta}$ is already written in terms of sine and cosine, so we appear to be stuck. However, multiply by 1, and force the denominator to be the difference of two squares.

$$\begin{aligned}
\frac{\cos\theta}{1-\sin\theta} &\equiv \frac{\cos\theta}{1-\sin\theta} \times \frac{1+\sin\theta}{1+\sin\theta} \\
&\equiv \frac{\cos\theta(1+\sin\theta)}{1^2-\sin^2\theta} \\
&\equiv \frac{\cos\theta(1+\sin\theta)}{\cos^2\theta} && \blacktriangleleft \textbf{ Using identity 1.} \\
&\equiv \frac{1+\sin\theta}{\cos\theta} \\
&\equiv \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} \\
&\equiv \sec\theta + \tan\theta
\end{aligned}$$

Note that $\cos\theta$ is a common factor in the numerator and the denominator.

So we arrive at the RHS, as required. That is,

$$\frac{\cos\theta}{1-\sin\theta} \equiv \sec\theta + \tan\theta$$

3.2 Equations and Identities

Exercise

Technique

1 Solve the following trigonometric equations for $0^\circ \le \theta \le 360^\circ$ (giving angles correct to one decimal place):

- a $3\sin\theta - 4\cos\theta = 0$
- b $\sin\theta + 2\cos\theta = 0$
- c $3\sin\theta + \cos\theta = 0$
- d $4\sin\theta = \cos\theta$
- e $\cot\theta = \tan\theta$
- f $2\sin\theta = \frac{1}{2}\operatorname{cosec}\theta$

1 b

2 Solve these equations completely for $-180^\circ \le \theta \le 180^\circ$:

- a $\tan^2\theta - 3\tan\theta + 2 = 0$
- b $3\sin^2\theta - 4\sin\theta + 1 = 0$
- c $2\cos^2\theta + \cos\theta - 1 = 0$
- d $4\cos^2\theta = 3$
- e $8\sin^2\theta - 6\sin\theta + 1 = 0$
- f $3\tan^2\theta = 1$

3 Solve these equations completely in the range $0^\circ \le \theta \le 360^\circ$ (giving answers correct to one decimal place where necessary):

- a $\sin 2\theta = \frac{1}{2}$
- b $\tan(\theta - 33^\circ) = 0.4816$
- c $\cos(2\theta - 20^\circ) = 0.212$
- d $\tan(3\theta - 40^\circ) = 6$
- e $\sin(\theta - 90^\circ) = 0.75$
- f $\tan 2\theta = 1$

3 a

4 Use Pythagorean identities to solve these equations in the range $-180^\circ \le \theta \le 180^\circ$:

- a $6\cos^2\theta + \sin\theta - 5 = 0$
- b $2\sin^2\theta + 5\cos\theta + 1 = 0$
- c $\sec^2\theta = 3 - \tan\theta$
- d $2\sec^2\theta = 5\tan\theta$
- e $\operatorname{cosec}^2\theta = 3\cot\theta - 1$
- f $\tan^2\theta + \sec^2\theta = 17$

5 Prove the following identities:

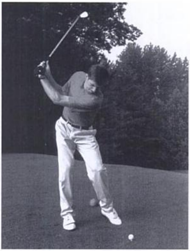
- a $\sec^2\theta + \operatorname{cosec}^2\theta \equiv \sec^2\theta\operatorname{cosec}^2\theta$
- b $\sec\theta + \tan\theta \equiv \dfrac{1+\sin\theta}{\cos\theta}$
- c $\cos^2\theta + 3\sin^2\theta \equiv 3 - 2\cos^2\theta$
- d $\cos^4\theta - \sin^4\theta \equiv \cos^2\theta - \sin^2\theta$
- e $\cot^2\theta + \cos^2\theta \equiv (\operatorname{cosec}\theta - \sin\theta)(\operatorname{cosec}\theta + \sin\theta)$
- f $\dfrac{\sin\theta\tan\theta}{1-\cos\theta} \equiv 1 + \sec\theta$

Contextual

1 The trajectory of a golf ball struck from a tee can be modelled by the equation

$$y = x\tan\theta - \frac{x^2\sec^2\theta}{80}$$

where θ is the acute angle of projection, x is the horizontal distance of the ball from the tee in metres and y is the vertical height of the ball above the fairway, also measured in metres. If the ball just clears a tree 16.8 m high, whose base is 16 m from the tee, find, to three significant figures, the angle of projection.



2 The equation of the path of a projectile referred to horizontal and upward vertical axes Ox, Oy for a golf ball is

$$y = x\tan\theta - \frac{x^2\sec^2\theta}{50}$$

Show how to reduce this equation to a quadratic equation in $\tan\theta$. Show that there are two distinct values of θ for which the ball can pass through a given point (X, Y), where $X > 0$, provided $200Y < 2500 - 4X^2$. Interpret this result in terms the golfer would understand.

3.3 Compound and Double Angle Formulas

In the trigonometry studied so far all the relationships contain trigonometric functions of a single variable, θ. There are other useful relationships that involve trigonometric functions of two variables and these are known as compound angle formulas. They are also known as the addition theorems since they show how the trigonometric functions of a sum or difference of two angles can be expressed in terms of the trigonometric functions of the individual angles.

$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$ Identity 1 ◀ **Learn these formulas.**

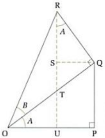
$\sin(A-B) \equiv \sin A \cos B - \cos A \sin B$ Identity 2

$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$ Identity 3

$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$ Identity 4

Notice that they are written as identities. This is because they are true for all values of A and B.

Identity 1 can be proved for the case A and B acute by considering a diagram illustrating the geometry of the angle sum $(A+B)$.



Let OPQ and OQR be right-angled triangles containing angles A and B respectively (as shown in the diagram). Look at triangle OQR.

Use the geometry of figure to show that $\angle$SRQ must be A; that is triangles OPQ and RSQ are similar.

If $OR = 1$ then $RQ = \sin B$ and $OQ = \cos B$.

Now look at triangle RQT.

$RQ = \sin B$, so $RS = \sin B \cos A$ (call this result [1]).

Now look at triangle OQR.

$QP = OQ \sin A$, and $OQ = \cos B$,

so $QP = \cos B \sin A$ (call this result [2]).

Now look at triangle ORU.

$$OR = 1, \text{ so } RU = \sin(A + B).$$

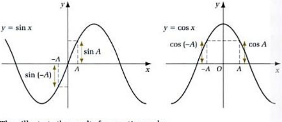
But $\quad RU = RS + SU = RS + QP$, so from results [1] and [2],

$$\sin(A + B) = \sin B \cos A + \cos B \sin A.$$

That is, $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

It is usual to write the expressions with the angles in alphabetical order.

This demonstrates a proof for A and B acute. The proof of $\cos(A + B)$ is very similar. The results for the differences, $\sin(A - B)$ and $\cos(A - B)$, can be found by replacing B with $-B$ in each of the proven results and using the results $\sin(-B) = -\sin B$ and $\cos(-B) = \cos(B)$. These last results can be seen clearly from the graphs of sine and cosine.



They illustrate the results for negative angles.

$\sin(-A) = -\sin A$ $\quad$ $\cos(-A) = \cos A$

The compound angle formulas hold for *all* angles.

These compound angle formulas allow us to evaluate more sines and cosines exactly and demonstrate further identities.

Example 1

a Find $\sin 15^\circ$ in surd form.
b Find the exact value of $\cos 75^\circ$.

Solution

a $\sin 15^\circ = \sin(60^\circ - 45^\circ)$

We know we want an answer in surd form, so try to write the angle as a sum of special angles.

$= \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ$ ◀ **Using identity 2.**

$= \left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2} \times \frac{1}{\sqrt{2}}\right)$

$= \frac{1}{2\sqrt{2}}(\sqrt{3} - 1)$

$= \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$

Remember to rationalise the denominator.

b $\cos 75^\circ = \cos(45^\circ + 30^\circ)$

$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}} \times \frac{1}{2}\right)$$

$$= \frac{1}{2\sqrt{2}}(\sqrt{3} - 1)$$

$$= \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$$

This is consistent with the result from Section 3.1, where $\cos\theta = \sin(90^\circ - \theta)$ when θ is acute.

These compound angle formulas also allow the addition theorems for tangent to be established.

$$\tan(A + B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$$ Identity 5 ◀ Learn these results.

$$\tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$ Identity 6

Example 2

Prove the identity $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

Solution

Beginning with the LHS,

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

Recalling the definition of tangent.

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

◀ Using identities 1 and 3.

$$= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

Dividing each term in both the numerator and denominator by $\cos A \cos B$, converts sines into tangents.

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Eliminate terms; write tangents where possible.

So we arrive at the RHS. The compound angle formula for $\tan(A - B)$ can be derived in the same way.

Example 3

Solve the equation $\sin(\theta + 45^\circ) = 2\cos(\theta + 45^\circ)$ for $0^\circ \leq \theta \leq 360^\circ$.

Solution

$$\sin(\theta + 45^\circ) = 2\cos(\theta + 45^\circ) \Rightarrow \frac{\sin(\theta + 45^\circ)}{\cos(\theta + 45^\circ)} = 2$$

$$\Rightarrow \quad \tan(\theta + 45^\circ) = 2$$

$$\Rightarrow \quad \frac{\tan\theta + \tan 45^\circ}{1 - \tan\theta \tan 45^\circ} = 2$$ ◀ Using identity 5.

$$\Rightarrow \quad \frac{\tan\theta + 1}{1 - \tan\theta} = 2$$

$$\Rightarrow \quad \tan\theta + 1 = 2 - 2\tan\theta$$

An alternative metho is to solve $\theta + 45^\circ = \tan^{-1}(2)$.

Remembering that 45 a special angle.

So $3\tan\theta = 1$, and $\tan\theta = \frac{1}{3}$.
The solutions are $\theta = 18.4^\circ$, 198.4° (1 d.p.).

Double angles

The compound angle formulas provide some more very useful results in the special case where $A = B$. These are known as double angle formulas.

$$\sin 2\theta = 2\sin\theta\cos\theta$$ ◀ Learn these results.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

To get these results simply put $A = B = \theta$ in the formulas for $\sin(A + B)$, $\cos(A + B)$ and $\tan(A + B)$. We can also use the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$, in the expression for $\cos 2\theta$ to provide two other versions.

$$\cos 2\theta = 2\cos^2\theta - 1$$ ◀ Learn these results.

$$\cos 2\theta = 1 - 2\sin^2\theta$$

Example 4

Find $\cos 2\theta$ when $\cos\theta = -\frac{1}{2}$.

Solution

Using $\cos 2\theta = 2\cos^2\theta - 1$,

$$\cos 2\theta = 2 \times (-\tfrac{1}{2})^2 - 1$$
$$= 2 \times \tfrac{1}{4} - 1 = -\tfrac{1}{2}$$

The result can be checked by using the methods of Section 3.1. When $\cos\theta = -\frac{1}{2}$, the cosine is negative so θ is in the second or third quadrant.

We know that $\cos 60^\circ = \frac{1}{2}$, because 60° is a special angle. So in the second quadrant, $\cos(180^\circ - 60^\circ) = -\frac{1}{2}$. That is, $\cos 120^\circ = -\frac{1}{2}$. So $\theta = 120^\circ$, and $2\theta = 240^\circ$.

Now $\cos 240^\circ = \cos(180^\circ + 60^\circ)$, and 240° is in the third quadrant, so $\cos 240^\circ = -\cos 60^\circ = -\frac{1}{2}$. So when $\cos\theta = -\frac{1}{2}$, $\cos 2\theta = -\frac{1}{2}$.

Example 5

Find the exact value of $\dfrac{2\tan 67\frac{1}{2}^\circ}{1-\tan^2 67\frac{1}{2}^\circ}$.

Solution

$$\begin{aligned}\frac{2\tan 67\frac{1}{2}^\circ}{1-\tan^2 67\frac{1}{2}^\circ} &= \tan(2\times 67\tfrac{1}{2}^\circ)\\ &= \tan 135^\circ\\ &= \tan(180^\circ - 45^\circ)\\ &= -\tan 45^\circ\\ &= -1\end{aligned}$$

Recall that $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$.

In the second quadrant, tangent is negative.

Example 6

Prove the identity $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$.

Solution

Beginning with the LHS,

$$\begin{aligned}\tan\theta + \cot\theta &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\\ &= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} \quad \blacktriangleleft \text{ Adding fractions.}\\ &= \frac{1}{\sin\theta\cos\theta}\\ &= \frac{2}{2\sin\theta\cos\theta}\\ &= \frac{2}{\sin 2\theta} \quad \blacktriangleleft \text{ Remember the definition of cosec } \theta.\\ &= 2\operatorname{cosec} 2\theta\end{aligned}$$

Remember the definition of $\cot\theta$.

$\sin^2\theta + \cos^2\theta = 1$.

Multiply numerator and denominator by 2 to create a double angle formula.

3.3 Compound and Double Angle Formulas

Exercise

Technique

1 Without using a calculator, find the exact value of the following:

Hint: use compound angle formulas

a	$\sin 75^\circ$	**b**	$\cos 105^\circ$	**c**	$\tan 15^\circ$
d	$\cos 15^\circ$	**e**	$\sin 105^\circ$	**f**	$\tan 105^\circ$

2 Write down the exact value of the following expressions:

a $\cos 40^\circ \cos 50^\circ - \sin 40^\circ \sin 50^\circ$ **d** $\sin 40^\circ \cos 20^\circ + \cos 40^\circ \sin 20^\circ$

b $\sin 60^\circ \cos 15^\circ - \cos 60^\circ \sin 15^\circ$ **e** $\cos 85^\circ \cos 25^\circ + \sin 85^\circ \sin 25^\circ$

c $\dfrac{\tan 47^\circ - \tan 17^\circ}{1 + \tan 47^\circ \tan 17^\circ}$ **f** $\dfrac{\tan 90^\circ + \tan 30^\circ}{1 - \tan 90^\circ \tan 30^\circ}$

2 c

3 Solve the following equations for $0^\circ \leq \theta \leq 360^\circ$, giving angles correct to one decimal place:

a $\sin(\theta + 30^\circ) = 2\cos\theta$ **b** $\sin(\theta + 15^\circ) = 3\cos(\theta - 15^\circ)$

c $\cos(\theta - 60^\circ) = \frac{1}{2}\sin\theta$ **d** $\sin(\theta + 45^\circ) = -2\cos\theta$

3 a

4 Prove the following identities:

a $\cos(45^\circ - \theta) - \cos(45^\circ + \theta) \equiv \sqrt{2}\sin\theta$

b $\tan(\theta + 45^\circ) + \tan(\theta - 45^\circ) \equiv 2\tan 2\theta$

c $\sin(A + B) + \sin(A - B) \equiv 2\sin A\cos B$

d $\tan A + \tan B \equiv \dfrac{\sin(A + B)}{\cos A\cos B}$

5 If $\sin A = \frac{3}{5}$, where A is acute, and $\cos B = \frac{5}{13}$, find the exact value of the following:

a	$\sin(A + B)$	**e**	$\tan(A + B)$	**i**	$\tan 2A$
b	$\cos(A + B)$	**f**	$\tan(A - B)$	**j**	$\sin 2B$
c	$\sin(A - B)$	**g**	$\sin 2A$	**k**	$\cos 2B$
d	$\cos(A - B)$	**h**	$\cos 2A$	**l**	$\tan 2B$

6 Prove the following identities:

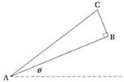
a $\sec\theta\,\mathrm{cosec}\,\theta \equiv 2\,\mathrm{cosec}\,2\theta$ **b** $\sec 2\theta + \tan 2\theta \equiv \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$

c $\dfrac{\sin 2\theta}{1 - \cos 2\theta} \equiv \cot\theta$ **d** $\dfrac{\sin 2\theta}{1 + \cos 2\theta} \equiv \tan\theta$

6 d

Contextual

1 **a** A ramp for a wheelchair, as illustrated in the diagram, can be moved by a person lifting on a handle at B and wheeling the ramp on a castor attached at A. The dimensions of the ramp are $AB = 3$ m and $BC = 50$ cm. The base of the ramp is inclined at an angle θ to the ground as shown.



i If the ramp is to be moved through a doorway 2 m high, explain why $\sin(\theta + 9.5^\circ) \leq 0.6576$.

ii Solve this equation for θ.

iii Comment on the practicalities of θ achieving its maximum value.

b A second ramp can similarly be moved by a person lifting on a handle at B. The ramp can be wheeled on a castor at A. The dimensions of this ramp are $AB = 2.5$ m and $BC = 60$ cm.

i Write down the new equation for θ.

ii Solve the equation.

iii Compare the heights to which B has to be lifted.

3.4 The Cosine Rule and the Sine Rule

'Solve a triangle' means find all the angles and the lengths of all the sides of the triangle. Given three of the six quantities (angles or sides) can the remaining three be found?

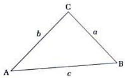
Pythagoras' theorem applies to right-angled triangles. How do we deal with non-right-angled triangles? One technique is to use the **cosine rule**.

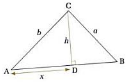
The cosine rule

For any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc\cos A$$

To demonstrate this result, drop a perpendicular from C to meet AB at D. Now look at triangle ACD.





Using Pythagoras' theorem,

$$b^2 = h^2 + x^2$$

So $h^2 = b^2 - x^2$ [1]

Now look at triangle BCD. Using the same technique,

$$a^2 = h^2 + (c - x)^2$$
$$h^2 = a^2 - (c - x)^2 \quad [2]$$

Eliminating h by equating [1] and [2],

$$b^2 - x^2 = a^2 - (c - x)^2$$

So $b^2 - x^2 = a^2 - (c^2 - 2cx + x^2)$ Multiplying out the bracket.

$$b^2 - x^2 = a^2 - c^2 + 2cx - x^2$$

$$b^2 = a^2 - c^2 + 2cx$$ The x^2 terms cancel each other out.

$$a^2 = b^2 + c^2 - 2cx$$

But from triangle ACD, $\cos A = \frac{x}{b}$, giving $x = b\cos A$.

So now $a^2 = b^2 + c^2 - 2bc\cos A$.

By using a symmetrical argument, the two other equivalent versions can be found (that is, by dropping the perpendicular from any angle, or simply by relabelling the triangle).

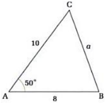
$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

So the cosine rule can be used to find the length of the third side, when the lengths of two sides in a triangle and the angle between them are known.

Example 1

In triangle ABC, $b = 10\,\text{cm}$, $c = 8\,\text{cm}$ and $A = 50^\circ$. Find a.



Solution

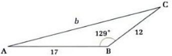
Using $a^2 = b^2 + c^2 - 2bc\cos A$,

$$\begin{aligned} a^2 &= 10^2 + 8^2 - 2 \times 10 \times 8 \times \cos 50^\circ \\ &= 100 + 64 - 160\cos 50^\circ \\ &= 164 - 102.85 \\ &= 61.15 \end{aligned}$$

$$\begin{aligned} \text{So } a &= \sqrt{61.15} \\ &= 7.82 \text{ cm (3 s.f.)} \end{aligned}$$

Example 2

In triangle ABC, $a = 12\,\text{cm}$, $c = 17\,\text{cm}$ and $B = 129^\circ$. Find b.



Solution

Although B is obtuse, the rule still applies.
Using $b^2 = a^2 + c^2 - 2ac\cos B$,

$$b^2 = 12^2 + 17^2 - 2 \times 12 \times 17 \times \cos 129^\circ$$
$$= 144 + 289 + 256.76$$
$$= 689.76$$
$$\text{So} \quad b = \sqrt{689.76}$$
$$= 26.3\,\text{cm (3 s.f.)}$$

There is a change of s because $\cos 129^\circ$ is negative.

The three cosine rule formulas can be rearranged to give expressions for $\cos A$, $\cos B$ and $\cos C$ in terms of a, b and c.

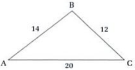
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Learn these results.

This means that if the lengths of each of the three sides are known then each of the three angles can be found.

Example 3

In triangle ABC if $a = 12\,\text{cm}$, $b = 20\,\text{cm}$ and $c = 14\,\text{cm}$, find A, B and C.



Solution

Using $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$,

$$\cos A = \frac{20^2 + 14^2 - 12^2}{2 \times 20 \times 14}$$
$$= \frac{400 + 196 - 144}{560} = \frac{452}{560} = 0.8071$$
$$\text{So} \quad A = \cos^{-1}(0.8071)$$
$$A = 36.2^\circ \text{ (1 d.p.)}$$

Using $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$,

$$\cos B = \frac{12^2 + 14^2 - 20^2}{2 \times 12 \times 14}$$

$$= \frac{144 + 196 - 400}{335} = -\frac{60}{336} = -0.1786$$

So $B = \cos^{-1}(-0.1786)$

$B = 100.3^\circ$ (1 d.p.)

The negative cosine implies that the angle is obtuse (second quadrant).

We know that the angles in a triangle add up to 180°. So now

$$C = 180^\circ - (A + B)$$

$$= 180^\circ - (36.2^\circ + 100.3^\circ)$$

$$= 180^\circ - 136.5^\circ$$

$$= 43.5^\circ \text{ (1 d.p.)}$$

The sine rule

The sine rule is another useful result to help solve triangles. For any triangle ABC,

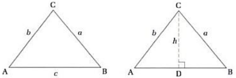
Unlike the cosine rule this doesn't look at all like Pythagoras' theorem.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This particular version of the sine rule is useful for finding sides. To find angles the more convenient form to use is

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

To demonstrate this result, drop a perpendicular from C to meet AB at D.



Looking at triangle ACD, $\frac{h}{b} = \sin A$.

So $h = b \sin A$

Looking at triangle BCD, $\frac{h}{a} = \sin B$.

So $h = a \sin B$

Now eliminate h by equating these two results.
Then $a \sin B = b \sin A$.

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

Use the geometry of the figure or simply relabel to find the equivalent ratio $\frac{c}{\sin C}$.

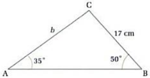
The sine rule is useful in the following situations:

- two (and hence three) angles and one side are known;
- two sides and one angle (but not the angle between them) are known;
- one angle and its opposite side, and one other piece of information are known.

Example 4

In triangle ABC, $A = 35°$, $a = 17$ cm and $B = 50°$. Find b.

Notice that we could find $C = 95°$ directly from the information given in the question using the angle sum triangle.



Solution

Using $\frac{a}{\sin A} = \frac{b}{\sin B}$,

$$\frac{17}{\sin 35°} = \frac{b}{\sin 50°}$$

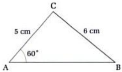
$$\text{So } b = \frac{17 \times \sin 50°}{\sin 35°}$$

$$= \frac{17 \times 0.7660}{0.5736}$$

$$b = 22.7 \text{ cm (1 d.p.)}$$

Example 5

In triangle ABC, $A = 60°$, $a = 6$ cm and $b = 5$ cm. Find B.



Solution

Using $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$,

$$\frac{\sin 60°}{6} = \frac{\sin B}{5}$$

$$\Rightarrow \sin B = \frac{5 \times 0.8660}{6}$$

$$= 0.7217$$

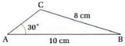
So $B = \sin^{-1}(0.7217) = 46.2°$

Use this form so that the unknown is in the numerator (hence simplifying the algebra).

Exercise some caution when using the sine rule to find angles. Sometimes an ambiguity may arise, where two possible angles will solve the trigonometric equation involving sine. The calculator will always give the acute angle but on occasions the obtuse angle gives an equally valid, if different, solution.

Example 6

Solve the triangle for which $A = 30°$, $a = 8$ cm and $c = 10$ cm.



Solution

Using the sine rule to find C,

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\frac{\sin C}{10} = \frac{\sin 30°}{8}$$

$$\sin C = \frac{0.5 \times 10}{8} = 0.625$$

So $C = 38.7°, 141.3°$ (1 d.p.)

Notice that the obtuse angle (141.3°) is possible here.

Angle B can now be found by using the angle sum of a triangle.

When $C = 38.7^\circ$, $B = 180^\circ - (30^\circ + 38.7^\circ) = 111.3^\circ$.
When $C = 141.3^\circ$, $B = 180^\circ - (30^\circ + 141.3^\circ) = 8.7^\circ$.

Now use the cosine rule (or the sine rule again) to find b for each of the possibilities for B.

Using the cosine rule, when $B = 111.3^\circ$,

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ &= 8^2 + 10^2 - 2 \times 8 \times 10 \times \cos 111.3^\circ \\ &= 164 + 58.12 \\ &= 222.12 \end{aligned}$$

So $\quad b = 14.9\,\text{cm}$ (1 d.p.).

Using the cosine rule, when $B = 8.7^\circ$,

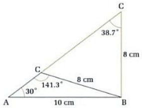
$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ &= 8^2 + 10^2 - 2 \times 8 \times 10 \times \cos 8.7^\circ \\ &= 164 - 158.16 \\ &= 5.84 \end{aligned}$$

So $\quad b = 2.4\,\text{cm}$ (1 d.p.)

This means the triangle ABC has two distinct solutions.

$A = 30^\circ$, $a = 8\,\text{cm}$	$A = 30^\circ$, $a = 8\,\text{cm}$
$B = 111.3^\circ$, $b = 14.9\,\text{cm}$	$B = 8.7^\circ$, $b = 2.4\,\text{cm}$
$C = 38.7^\circ$, $c = 10\,\text{cm}$	$C = 141.3^\circ$, $c = 10\,\text{cm}$

One sketch can be used to illustrate the geometrical interpretation.



3.4 The Cosine Rule and the Sine Rule

Exercise

Technique

1 Use the cosine rule to find the length of b in the following triangles:

a $B = 70^\circ$, $a = 8\,\text{cm}$, $c = 12\,\text{cm}$
b $B = 120^\circ$, $a = 11\,\text{cm}$, $c = 15\,\text{cm}$
c $B = 48^\circ$, $a = 14.2\,\text{cm}$, $c = 8.5\,\text{cm}$

1 a

In this exercise give lengths of sides and sizes of angles correct to 1 d.p.

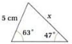
2 Use the cosine rule to find the angles in the following triangles:

a $a = 6\,\text{cm}$, $b = 4\,\text{cm}$, $c = 5\,\text{cm}$
b $a = 13\,\text{cm}$, $b = 12\,\text{cm}$, $c = 19\,\text{cm}$
c $a = 7\,\text{cm}$, $b = 8\,\text{cm}$, $c = 14\,\text{cm}$.

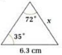
3 Calculate the length of the lettered side in the following triangles:

3 a

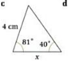
a



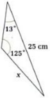
b



c

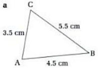


d

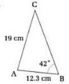


4 Solve the following triangles:

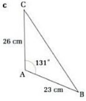
a



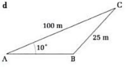
b



c



d



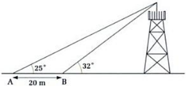
Contextual

1 A ship sails 4 nautical miles from A to B on a bearing of 075°. It then changes to a bearing of 150° and sails 6 nautical miles to C. Calculate the distance *AC* and the bearing of C from A.

2 Ship A is 12 miles from lighthouse B on a bearing of 050°. A second ship C is 15 miles from the same lighthouse on a bearing of 330°. Find the distance between A and C and the bearing of C from A.

3 A, B and C are three triangulation points used on an Ordnance Survey map. B is 5 km due south of A. C is 4 km from B on a bearing of 300°. Find the distance from A to C and the bearing of C from A.

4 From a point A on the same level as the base of a radio mast, the angle of elevation of the top of the mast is 25°. From a point B, 20 metres closer to the mast, and on the same level, the angle of elevation is 32°. Find the height of the radio mast.



3.5 The Area of a Triangle

You should already be familiar with the rule for the area of a triangle that reads:

area $= \frac{1}{2} \times$ base $\times$ perpendicular height

This formula is only useful if the length of the 'base' and the 'perpendicular height' of the triangle are known. When this is not the case other techniques are required.

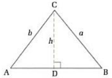
A more useful result is one that incorporates lengths of sides and sizes of angles. In general the area of a triangle ABC is given by:

area $= \frac{1}{2}bc\sin A = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B$ ◀ Learn these results.

One way of remembering these formulas is that they include two sides and the angle between them.

Remember the symmetry of this expression.

To demonstrate these results drop a perpendicular from A, B or C to meet a, b or c respectively. Suppose a perpendicular that meets AB at D is drawn, as shown.



Looking at triangle ACD, $\frac{h}{b} = \sin A$.

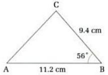
$$\begin{aligned}\text{area of triangle ACB} &= \tfrac{1}{2} \times \text{base} \times \text{perpendicular height} \\ &= \tfrac{1}{2} \times c \times h \\ &= \tfrac{1}{2} \times c \times b \times \sin A \\ &= \tfrac{1}{2}bc\sin A\end{aligned}$$

To show one of the other symmetrical versions, simply drop perpendicular lines onto BC or AC.

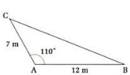
Example 1

Find the areas of the following triangles:

a



b



Solution

a Using area $= \frac{1}{2}ac\sin B$,

$$\text{area} = \tfrac{1}{2} \times 9.4 \times 11.2 \times \sin 56^\circ$$
$$= 43.6\,\text{cm}^2 \text{ (3 s.f.)}$$

b Using area $= \frac{1}{2}bc\sin A$,

$$\text{area} = \tfrac{1}{2} \times 7 \times 12 \times \sin 110^\circ$$
$$= 39.5\,\text{m}^2 \text{ (3 s.f.)}$$

Hero's formula

These formulas for area are not really useful if only the lengths of the sides of the triangle are known. The cosine rule could be used to find an angle but often it is much more efficient to use another method called Hero's formula. This is named after Heron of Alexandria (who lived about 75BC). Hero's formula has many practical applications. For example surveyors who know lengths of sides of a three-sided lot can easily compute the area.

In the great city of Alexandria, mathematicians were forced to consider the of calculation by probl encountered during the study of astronomy an mechanics. This is an example of where the need to solve problems forced the development new techniques in mathematics.

Hero's formula states that, for any triangle ABC with sides of length a, b and c:

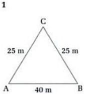
area $= \sqrt{s(s-a)(s-b)(s-c)}$

where s is the semi-perimeter, $s = \dfrac{a+b+c}{2}$. ◄ **Learn this result.**

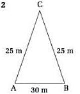
Example 2

Which of the following triangles has the larger area?

1



2



Solution

Looking at triangle **1**,

$$s = \tfrac{1}{2}(25 + 25 + 40) = 45$$

So the area of triangle **1** $= \sqrt{45(45-25)(45-25)(45-40)}$

$$= \sqrt{45 \times 20 \times 20 \times 5}$$

$$= \sqrt{90\,000}$$

$$= 300\,\text{m}^2$$

Looking at triangle **2**,

$s = \frac{1}{2}(25 + 25 + 30) = 40$

So the area of triangle **2** $= \sqrt{40(40-25)(40-25)(40-30)}$

$$= \sqrt{40 \times 15 \times 15 \times 10}$$

$$= \sqrt{90\,000}$$

$$= 300\,\text{m}^2$$

Since the area of both triangles **1** and **2** is $300\,\text{m}^2$, neither has a larger area.

3.5 The Area of a Triangle

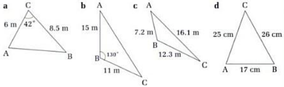
Exercise

Technique

1 For a right-angled triangle with sides 3 m, 4 m and 5 m find the area using:

- **a** area $= \frac{1}{2}$(base × perpendicular height)
- **b** area $= \frac{1}{2}ab\sin C$
- **c** Hero's formula.

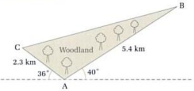
2 Find the areas of the following triangles:



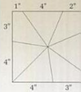
Contextual

1 The area of an acute-angled triangle ABC is $12\,\text{cm}^2$. Given that $a = b = 5$ cm, find c.

2 A surveyor wishes to estimate the area of a triangular patch of woodland. A sketch is made showing the following measurements. Estimate the area of the woodland.



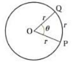
3 Paula, the mathematician, wants to share her square birthday cake equally amongst the seven people at her party. She wants everyone to have the same amount of cake and icing. Her friend, Kevin, suggests dividing the perimeter by 7 and cutting from the centre to the appropriate points on the perimeter. If the cake measures 7 inches down each side and is 3 inches high will Kevin's suggestion work?



3.6 Radian Measure

All methods of measuring angles are based on division of the circle. We have previously measured angles in degrees, where 360° is a full circle. So a degree is simply $\frac{1}{360}$ of a circle. Another way of measuring angles is to compare the length of an arc formed by the angle with the radius of the circle. The unit used in this method is called the **radian**.

In the diagram the radius of the circle is r and the length of the arc PQ is also r.
The angle θ is said to be 1 radian.



An alternative notation for radian measure is the use of a lower case c: 1 rad = 1^c.

$$\frac{\text{arc length}}{\text{radius}} = \frac{r}{r} = 1$$

The circumference of the circle is $2\pi r$. This means that a complete circle, defined in degrees as 360°, is 2π in radians.

$$\frac{\text{circumference}}{\text{radius}} = \frac{2\pi r}{r} = 2\pi$$

So 360° = 2π radians (rad).
Then 180° = π rad, so

Halving both sides.

$$1^\circ = \frac{\pi}{180}\ \text{rad}$$

Similarly, a radian has an equivalence in degrees.
We have 2π rad = 360°, so

$$1\ \text{rad} = \frac{360^\circ}{2\pi}$$

$$1\ \text{rad} = \frac{180^\circ}{\pi}$$

Using a calculator to evaluate this statement,

$$1\ \text{rad} = 57.29578\ldots^\circ$$
$$= 57.3^\circ\ (3\ \text{s.f.})$$

Since this isn't a rational value (because of the involvement of π, which is irrational) it is more usual to express radians as fractions of π.

Example 1

Express the special angles 30°, 45°, 60° and 90° as radians.

Solution

Since $1^\circ = \dfrac{\pi}{180}$ rad,

$$30^\circ = 30 \times \frac{\pi}{180} = \frac{\pi}{6}\text{ rad} \qquad 60^\circ = 60 \times \frac{\pi}{180} = \frac{\pi}{3}\text{ rad}$$

$$45^\circ = 45 \times \frac{\pi}{180} = \frac{\pi}{4}\text{ rad} \qquad 90^\circ = 90 \times \frac{\pi}{180} = \frac{\pi}{2}\text{ rad}$$

Learn these results.

Example 2

Convert $\frac{5\pi}{6}$ rad to degrees.

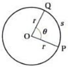
Solution

$$1\text{ rad} = \frac{360^\circ}{2\pi} \Rightarrow \frac{5\pi}{6}\text{ rad} = \frac{5\pi}{6} \times \frac{360^\circ}{2\pi}$$
$$= 150^\circ$$

In multiplying the fractions, π cancels.

Radian measure has several advantages in mathematics and will be used later in curve sketching and integration. This measure allows both axes to be scaled in real numbers and is particularly advantageous when the requirement is to solve a problem by reflecting in the line $y = x$, or finding the intersection of graphs.

Another application of radians is in finding arc lengths and sector areas. If we take a sector of a circle its arc length and area will be a fraction of the whole circle. The formulas for these have a simpler form when the sector angle is measured in radians.



When θ is in radians,

arc length, $s = r\theta$

sector area, $A = \frac{1}{2}r^2\theta$ ◀ **Learn these results.**

Example 3

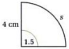
Find the arc length and area of a sector of radius 4 cm and angle 1.5 rad.

Solution

When θ is measured in radians,

arc length $s = r\theta = 4 \times 1.5 = 6\text{ cm}$

sector area $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 4^2 \times 1.5 = 12\text{ cm}^2$



Example 4

A piece of wire 40 cm long is bent into the shape of a sector. If the area of this sector is $100\,\text{cm}^2$ find:

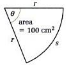
a an expression for the sector angle θ in terms of radius r

b the value of r

c the value of θ.

Solution

a Think carefully about the given information. The three lengths (both radii and the arc length) add up to 40 cm. So arc length $s = 40 - 2r$. Also, $s = r\theta$.



So $r\theta = 40 - 2r$

$$\theta = \frac{40 - 2r}{r}$$

b Now use the information about the area.

$$A = \tfrac{1}{2}r^2\theta$$

Substituting for θ from **a**,

$$100 = \tfrac{1}{2} \times r^2 \times \left(\frac{40 - 2r}{r}\right)$$

$$100 = \frac{r}{2} \times (40 - 2r)$$

Eliminating r from the denominator.

$$100 = \frac{r}{2} \times 2(20 - r)$$

Taking out the common factor, 2.

$$100 = 20r - r^2$$

So $r^2 - 20r + 100 = 0$

Form and solve the quadratic equation.

$$\Rightarrow (r - 10)(r - 10) = 0$$

$$(r - 10)^2 = 0$$

So $r = 10$. That is, the radius is 10 cm.

c $\theta = \dfrac{40 - 2r}{r} = \dfrac{40 - (2 \times 10)}{10} = 2$

So the sector angle is 2 rad.

3.6 Radian Measure

Exercise

Technique

1 Without using a calculator convert the following angles into radians, leaving the answer in terms of π:

a 15°	**c** 75°	**e** 225°
b 150°	**d** 180°	**f** 315°

2 Convert the following to radians, writing your answers correct to three significant figures:

a 20°	**c** 129°	**e** 269°
b 78°	**d** 222°	**f** 351°

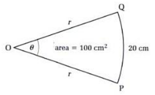
3 Without using a calculator convert the following angles to degrees:

a π	**c** 3π	**e** $\frac{\pi}{8}$
b $\frac{3\pi}{4}$	**d** $\frac{4\pi}{15}$	**f** $\frac{7\pi}{8}$

4 Find the arc lengths and sector areas of the following sectors, correct to one decimal place:

a $\theta = 1\,\text{rad}, r = 5\,\text{cm}$	**b** $\theta = 1.5\,\text{rad}, r = 8\,\text{cm}$
c $\theta = 2\,\text{rad}, r = 11\,\text{cm}$	**d** $\theta = 2.6\,\text{rad}, r = 9\,\text{cm}$

5 A sector OPQ has arc length 20 cm and sector area 100 cm^2 (as shown). Write down the two equations involving r and θ (the radius and sector angle). Solve these equations to find the value of r and θ. 5



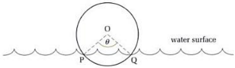
6 Given that the area of sector OPQ is 50 cm^2 and that $\theta = 1.2$ rad, find the radius of the circle.

7 Now go back to Exercise 3.2 on page 105 and do questions 1, 2, and 4 using radians. 6

Contextual

1 A wedge of cheese, 0.5 cm thick, has a cross-section that is a sector of a circle. If the radius of the circle is 5 cm and the arc length is 4 cm, find the volume of the cheese.

2 An oil drum of diameter 80 cm is floating, as shown.



a If the arc length PQ is 90 cm, find θ (in radians).

b Find the area of the minor sector OPQ.

c Use the cosine rule to work out the length of the chord PQ.

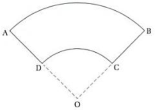
d Find the area of the triangle OPQ.

e If the drum is 1.2 m long, find the volume of the drum lying below the water surface level, correct to three significant figures.

Remember to put your calculator in RADIAN mode.

3 A circular cone with base radius r and slant height l is unrolled into a circular sector. Find the angle of this sector in radians. Use your answer to explain why the curved surface area of the cone is πrl.

4 A windscreen wiper clears a region ABCD of a car windscreen (assumed to be flat), where AB and DC are circular arcs centred at O. Given that $OD = 20$ cm, $DA = 50$ cm and $\angle AOB = 2.3$ rad, calculate the area of region ABCD.



(NEAB)

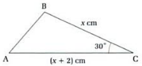
Consolidation

Exercise A

1 Find the four solutions of the equation $\sin 2\theta = \cos^2\theta$ in the interval $0^\circ < \theta < 360^\circ$. Give each of these solutions correct to the nearest degree.

(NEAB)

2 The diagram shows a triangle ABC in which angle $C = 30^\circ$, $BC = x$ cm and $AC = (x + 2)$ cm. Given that the area of triangle ABC is 12 cm^2, calculate the value of x.



(UCLES)

3 Figure 1 shows the points A and B which two ships have reached after leaving a harbour H. The points A and B are at distance of 8 km and 11 km from H respectively. The bearings of A and B from H are 048° and 120° respectively. Calculate:

a the distance between A and B, giving your answer to 0.1 km;
b the bearing of B and A, giving your answer to the nearest degree.

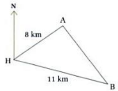


Figure 1

(ULEAC)

4 Solve the equation $4\cot^2\theta + 12\operatorname{cosec}\theta + 1 = 0$, giving all values of θ to the nearest degree in the interval $0^\circ \leq \theta \leq 360^\circ$.

(AEB)

5 Find, in degrees, the solution of the equation $\tan(2x + 30^\circ) = -\sqrt{3}$, for $0^\circ \leq x \leq 360^\circ$.

(NICCEA)

6 The angles x and y are acute.

a If $\cos 2x$ is negative, explain why you can say that $x > 45°$.
b If $\sin 4y$ is negative and $\cos 4y$ is positive, what value must y exceed?
c If the conditions in **a** and **b** remain true, state, with reasons, whether the following are positive or negative:
i $\tan(x + y)$
ii $\tan 3y$.

(MEI)

7 **a** Use the formula $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ to show that $\tan(\pi/12)$ is a root of the cubic equation $x^3 - 3x^2 - 3x + 1 = 0$.
b Solve the equation $\sin 2y = 2 - 2\cos 2y$ $(0 \le y \le \pi)$.

(OCSEB)

8 Show that $(\operatorname{cosec} x - 1)(\operatorname{cosec} x + 1)(\sec x - 1)(\sec x + 1) \equiv 1$.

(UCLES)

9 A circle centre O has an arc AB of length 13.44 cm and $\angle AOB = 1.6$ rad.

a Calculate the radius of the circle.
b Find the area of the region enclosed by the arc AB and the chord AB.

(WJEC)

10 **a** Determine, in degrees, the solutions of the equation $\tan x = 5$ for which $0° \le x \le 360°$, giving your answers to the nearest tenth of a degree.
b Determine, in radians, the solutions of the equation $3\cos^2 y + 8\sin y = 0$ for which $0 \le y \le 2\pi$, giving your answers to two decimal places.

(ULEAC)

Exercise B

1 Solve the following for x, y, z and t, giving all the values from $0°$ to $360°$ inclusive:

a $\sin(x/2) = 0.7$ **b** $\sec 2y = 1.5$
c $3\tan z = 5\sin z$ **d** $\sin^2 t = 1 + \cos t$

(OCSEB)

2 **a** Square both sides of the equation $\sin\theta - \cos\theta = \sin 2\theta$ and rearrange the resulting equation as a quadratic in $\sin 2\theta$.

b Deduce that $\sin 2\theta = \frac{1}{2}(\sqrt{5} - 1)$.

c Find the two solutions of the equation in **b** that lie in the range $0^\circ < \theta < 180^\circ$ and hence find the smallest positive value of θ that satisfies the equation given in **a**.

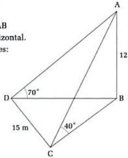
(WJEC)

3 A tetrahedron is as shown in the sketch ($AB = 12$ m, $DC = 15$ m, $\angle ACB = 40^\circ$, $\angle ADB = 70^\circ$), with AB vertical and the triangle DBC horizontal. Find, correct to two decimal places:

a the length of AC

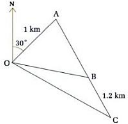
b the length of AD

c the angle $\angle DAC$.



(NICCEA)

4 From a coastguard station at O, three buoys, A, B and C, can be seen out at sea. A, B and C lie on a straight line such that the bearing of both B and C from A is 135°. The bearings of A and B from O are 030° and 100° respectively. The bearing of A is marked in the diagram, $OA = 1$ km, and $BC = 1.2$ km.



a State the angles of triangle OAB.

b Calculate the distance OB, giving your answer in kilometres correct to three significant figures.

c Calculate the distance OC and the bearing of C from O.

(OCSEB)

5 Express $\sin 4\theta$ in terms of $\sin 2\theta$ and $\cos 2\theta$, and hence express $\dfrac{\sin 4\theta}{\sin \theta}$ in terms of $\cos \theta$ only.

(UCLES)

6 a Show that $4(1+\cos\theta+\cos 2\theta)-3(\sin\theta+\sin 2\theta)$ may be written in the form $(1+2\cos\theta)(a\cos\theta+b\sin\theta)$, where a and b are integers to be determined.

Hence solve the equation $4(1+\cos\theta+\cos 2\theta)-3(\sin\theta+\sin 2\theta)=0$, giving all solutions between 0° and 360°.

b Given that $\tan(\alpha-\pi/4)=\frac{71}{97}$, find the exact values of $\tan\alpha$ and $\cos\alpha$. In triangle ABC, $AB = 25$ cm, $AC = 17$ cm and angle BAC $=\alpha$. Find the length of BC.

(WJEC)

7 Figure 2 shows an equilateral triangle ABC whose vertices lie on a circle, centre O, of radius r.

a Show that the length of a side of this triangle is $r\sqrt{3}$.

b Show that the ratio of the area of the shaded region to the area of the triangle is $(4\pi\sqrt{3}-9):9$.

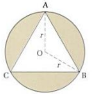


Fig. 2

(ULEAC)

Applications and Activities

All methods of measuring angles are based on division of a circle. Degrees are $\frac{1}{360}$ of a circle and can be subdivided into sixty minutes and then each minute subdivided into sixty seconds. Many calculators have a degree mode (DEG is usually displayed on the screen) and a key $^{\circ}\,'\,''$. The degree mode is usually located close to radian mode (RAD) and one other system of measuring angles; GRAD.

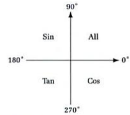
1 Investigate the GRAD system of measuring angles.

2 Find out what GRAD means and draw the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ when θ is measured in 'grads'.

3 What applications are there in everyday life with this scale?

Summary

- The trigonometric ratios are positive for angles in the following quadrants.

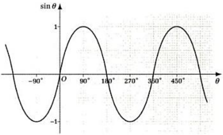


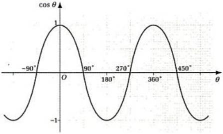
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

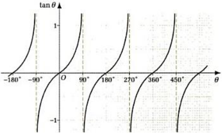
- The exact values of the trigonometric ratios of the special angles are:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

- The graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ have periods 360°, 360° and 180° respectively, and look like:







- The reciprocal ratios are:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

- The Pythagorean identities are:

$$\sin^2\theta + \cos^2\theta \equiv 1$$
$$\tan^2\theta + 1 \equiv \sec^2\theta$$
$$1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$$

- The compound angle formulas are:

$$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$$
$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

- The double angle formulas are:

$$\sin 2A \equiv 2\sin A\cos B$$
$$\cos 2A \equiv \cos^2 A - \sin^2 A$$
$$\equiv 2\cos^2 A - 1$$
$$\equiv 1 - 2\sin^2 A$$
$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

- The cosine rule, in its various forms is:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

- The sine rule is:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

- The area of a triangle can be calculated using

$$\text{area} = \tfrac{1}{2} \times \text{base} \times \text{perpendicular height}$$
$$\textbf{area} = \tfrac{1}{2}ab\sin C$$

Hero(n)'s formula: $\textbf{area} = \sqrt{s(s-a)(s-b)(s-c)}$

where $s = \tfrac{1}{2}(a + b + c)$.

- $1 \text{ radian} = \dfrac{360^\circ}{2\pi}$.

- In radian form, arc length $s = r\theta$ and sector area $A = \tfrac{1}{2}r^2\theta$

4 Functions

What you need to know

- How to change the subject of an equation.
- How to 'complete the square' for a quadratic expression.
- How to solve quadratic equations using either the factorisation method or the quadratic formula.
- How to express simple improper algebraic fractions as mixed fractions.

Review

1 Make x the subject of each of the following equations:

a $5x + 2y = 1$

b $y = 3x^2 - 2$

c $y = 4\sqrt{x - 1}$

d $y = \sqrt[3]{x} + 7$

e $y = 3 - \dfrac{10}{x}$

f $y = \dfrac{2x + 1}{x}$

2 Write each of the following expressions in the form $(x + a)^2 + b$ or $c(x + a)^2 + b$, where a, b and c are constants:

a $x^2 + 8x + 5$

b $x^2 - 4x + 3$

c $x^2 + x + 1$

d $x^2 - 3x - 5$

e $2x^2 + 12x$

f $3x^2 - 6x + 7$

3 Find the exact solutions of the following quadratic equations:

a $x^2 + 5x - 24 = 0$

b $x^2 - 5 = 0$

c $2x^2 - 14x + 20 = 0$

d $x^2 + 4x - 3 = 0$

e $x^2 + 6x - 1 = 0$

f $2x^2 - 6x + 1 = 0$

4 Rearrange each of the following into expressions that eliminate improper algebraic fractions:

a $\dfrac{x + 2}{x + 1}$

b $\dfrac{x + 1}{x + 2}$

c $\dfrac{x + 3}{x - 5}$

d $\dfrac{2x + 7}{x + 3}$

e $\dfrac{4x + 5}{2x - 1}$

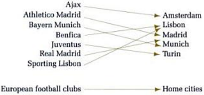
f $\dfrac{3x}{1 - x}$

An improper algebraic fraction is one where the highest power in the numerator is the same as or larger than the highest power in the denominator.

4.1 Mappings and Functions

Mappings

A **mapping** is a rule that relates one set of items or numbers to another. For example, the diagram below shows the mapping of seven leading European football clubs to their home cities. The set of inputs for a mapping is called the **domain**, and a set of outputs from the mapping is called the **co-domain**. In this example, the set of clubs is the domain and the set of cities is the co-domain.

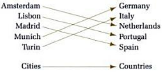


This relationship between the football clubs and their home cities is an example of a **many-to-one mapping**. Although each club obviously has only one home city, more than one club or **element** in the domain may map onto any given city in the co-domain. For example, Benfica and Sporting Lisbon are both based in the Portuguese capital, Lisbon.

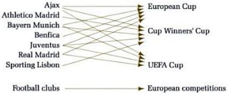
If we reverse this relationship, and map the cities to their football clubs we obtain a **one-to-many mapping**, as shown below. This shows that it is possible for an element in the domain to map to more than one element in the co-domain.

Two other types of mapping exist:

- one-to-one mappings
- many-to-many mappings.



The relationship between these five cities and their countries is a **one-to-one mapping**. Each city in the domain maps to one and only one country in the co-domain.



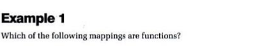
The relationship between the seven football clubs and the major European club competitions is an example of a **many-to-many mapping**. Several of these clubs have won more than one of these competitions.

Functions

Any mathematical mapping that takes any object in the domain and maps it to one, and only one, element in the co-domain is called a **function**. Since there can only be one possible image for each object in the domain, only many-to-one and one-to-one mappings are functions.

Example 1

Which of the following mappings are functions?



Solution

a This one-to-one mapping is a function. Each object in the domain maps to one, and only one, image.

b This many-to-one mapping is also a function. Note that it is not necessary for every element in the co-domain to be an image of an object in the domain.

c This many-to-many mapping is not a function because the element 'a' in the domain maps to more than one element in the co-domain.

The domain of a function is the set of **objects** or **input values** for which the function is defined. In Example 1 the set $\{a, b, c\}$ is the domain in functions **a** and **b**. The co-domain is a set containing the possible output values of a function. In Example 1 the set $\{x, y, z\}$ is the co-domain in functions **a** and **b**.

The **range** of a function is the set of **images**, or **output values**, to which the objects in the domain map. This can be the entire co-domain, as in the function in Example **1a**, which has the range $\{x, y, z\}$, or it can be a subset of the co-domain, as in the function in Example **1b**, which has the range $\{x, z\}$.

The words *onto* and *into* are used to highlight the difference. The domain of the function in Example **1a** maps *onto* its co-domain, because it uses every element in the co-domain. The domain of the function in Example **1b** maps *into* its co-domain, because only some of the elements are used.

A number of different but equivalent notations are used to define a function. For example, the function f, which maps values of x in the domain to values of $2x$ in the co-domain, may be written

$f: x \to 2x$ ◀ **f maps x to $2x$.**

or $f(x) = 2x$

or $y = 2x$

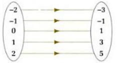
In this last form of notation, objects x in the domain map to images y in the co-domain. These (x, y) pairs can be used to represent the function graphically.

Example 2

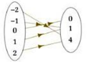
Functions f and g are defined for domain $\{-2, -1, 0, 1, 2\}$ by $f: x \to 2x + 1$ and $g: x \to x^2$. Draw a mapping diagram for each function. State the range of each function and explain which type of mapping is shown.

f maps x to $2x + 1$
g maps x to x^2

Solution



$f: x \rightarrow 2x + 1$



$g: x \rightarrow x^2$

The range of function f, for the given domain, is $\{-3, -1, 1, 3, 5\}$.
$f: x \rightarrow 2x + 1$ is a one-to-one mapping.

The range of function g, for the given domain, is $\{0, 1, 4\}$.
$g: x \rightarrow x^2$ is a many-to-one mapping.

It has been possible to draw mapping diagrams for each function in Example 2 because the domain is **discrete**. This means the domain consists of separate values of x, and does not contain any values between those stated.

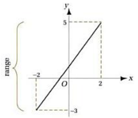
If the domain isn't discrete then it can contain any value within specified limits. In this case the domain is said to be **continuous**. Functions with continuous domains are usually represented graphically.

Example 3

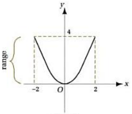
The functions f and g in Example 2 are defined for the continuous domain $\{x \in \mathbb{R}: -2 \leq x \leq 2\}$. In each case, sketch the graph of the function, and state the range.

$\{x \in \mathbb{R}: -2 \leq x \leq 2\}$ means all the real numbers between -2 and 2 (inclusive).

Solution



$f: x \rightarrow 2x + 1$



$g: x \rightarrow x^2$

The range of function f, for the given domain, is $\{y \in \mathbb{R}: -3 \leq y \leq 5\}$.

This means that the range of f is all the real numbers between -3 and 5 inclusive.

The range of function g, for the given domain, is $\{y \in \mathbb{R}: 0 \leq y \leq 4\}$.

All **linear** functions can be written in the form $f: x \to ax + b$, where a and b are constants. If a and b are both unknown, then they can be calculated using the value of the function at two different values of x.

Example 4

The linear function $f: x \to ax + b$ is such that $f(1) = 8$ and $f(3) = 14$. Find the values of a and b.

Solution

$f(1)$ and $f(3)$ are found by substituting $x = 1$ and $x = 3$ into the function.

$$f(1) = 8 \Rightarrow a + b = 8 \qquad [1]$$

$$f(3) = 14 \Rightarrow 3a + b = 14 \qquad [2]$$

We now have two lin simultaneous equatio

Eliminating b by subtracting equation [1] from equation [2],

$$2a = 6$$

$$\Rightarrow a = 3$$

Substituting $a = 3$ into equation [1],

$$3 + b = 8$$

$$\Rightarrow b = 5$$

So $a = 3$, $b = 5$ and the function is $f: x \to 3x + 5$.

All quadratic functions can be written in the general form $f: x \to ax^2 + bx + c$, where $a \neq 0$. If the coefficients a, b and c are all unknown, they can be found in a similar way to that used for linear functions (Example 4). This time use the value of the function f at three different values of x. This creates three linear simultaneous equations involving a, b and c. These can then be solved using either the elimination or substitution methods.

4.1 Mappings and Functions
Exercise
Technique

1 The functions below are defined for the given discrete domains. In each case:

i draw a mapping diagram
ii state the range of the function
iii state the type of mapping.

a $f: x \to 1 - 2x$ for $\{x: -10, -5, 0, 5, 10\}$
b $f: x \to 2x^2$ for $\{x: -2, -1, 0, 1, 2\}$
c $f: x \to +\sqrt{x}$ for $\{x: 0, 1, 4, 9, 16\}$
d $f: x \to \frac{2}{x}$ for $\{x: 1, 2, 3, 4, 5\}$

2 The functions below are defined for the given continuous domains. In each case:

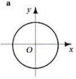
i sketch the graph of the function
ii state the range of the function
iii state the type of mapping.

a $f: x \to \frac{1}{2}x + 3$ for $\{x \in \mathbb{R}: \ 2 \le x \le 6\}$
b $f: x \to x^2 - 1$ for $\{x \in \mathbb{R}: -3 \le x \le 3\}$
c $f: x \to x^3$ for $\{x \in \mathbb{R}: -2 \le x \le 2\}$
d $f: x \to \frac{1}{x}$ for $\{x \in \mathbb{R}: \ 0 < x \le 3\}$

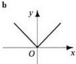
3 State whether or not each of the following graphs represents a function, giving reasons for your answer:

3

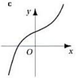
a



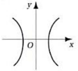
b



c



d



4 The mappings f and g are defined by:

$$f: x \to \begin{cases} 5 - x & \text{for } x \le 2 \\ \frac{1}{2}x^3 & \text{for } x \ge 2 \end{cases} \qquad g: x \to \begin{cases} 12x & \text{for } x \le 4 \\ 3x^2 & \text{for } x \ge 4 \end{cases}$$

Explain why g is a function, but f is not a function.

5 The functions f and g are such that $f(x) = 13x - 4$ and $g(x) = 4x^2 - 1$.

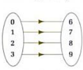
a Find the value of a for which $f(a) = 35$.
b Find the values of b for which $g(b) = 15$.
c Find the values of c for which $f(c) = g(c)$.

6 The linear function f is defined by $f: x \to ax + b$, where a and b are constants. Given that $f(2) = 3$ and $f(-3) = 13$, find a and b. Hence, calculate the value of c for which $f(c) = 0$.

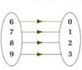
7 The quadratic function g is defined by $g: x \to ax^2 + bx + c$, where a, b and c are constants. Given that $g(0) = -3$, $g(1) = 4$ and $g(2) = 15$, find a, b and c. Hence calculate the values of d for which $g(d) = 0$.

4.2 Inverse Functions

For many mathematical mappings, there is a corresponding mapping that has the opposite or reversing effect. For example, the mapping 'subtract 6', or $x \to x - 6$ is the opposite of the mapping 'add 6', or $x \to x + 6$. For the domain $\{0, 1, 2, 3\}$, this last mapping has the range $\{6, 7, 8, 9\}$.



$x \to x + 6$



$x \to x - 6$

Taking $\{6, 7, 8, 9\}$ as the domain for the reverse mapping, $x \to x - 6$, this maps back onto the original domain.

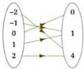
So for this particular domain, the mapping $x \to x + 6$ and its inverse $x \to x - 6$ are both one-to-one relationships. They are therefore both functions.

To write this more formally, if the function f is defined such that $f: x \to x + 6$, then its inverse function f^{-1} is defined as $f^{-1}: x \to x - 6$.

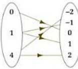
When working with functions, f^{-1} means 'the inverse of function f', and not 'the reciprocal of function f'.

Now consider the function $f(x) = x^2$, defined for the domain $\{x \in \mathbb{Z}: -2 \leq x \leq 2\}$. For this particular domain, the mapping of $x \to x^2$ is 'many-to-one'. Therefore $f(x)$ is a function. However, the reverse mapping is one-to-many. There is no unique relationship between the objects and images of this mapping. So, by definition, it cannot be a function.

Recall that $\mathbb{Z}$ is the set of all integers.



$f: x \to x^2$

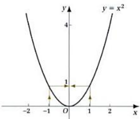


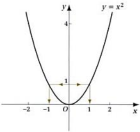
$f: x \to \pm\sqrt{x}$

Only those functions which, for a given domain, are one-to-one mappings, have inverse functions. This point can be illustrated graphically by changing the domain over which the function $f: x \to x^2$ is defined to $\{x \in \mathbb{R}: -2 \leq x \leq 2\}$. Find $f(-1)$ and $f(1)$. What do you notice? Both $f(-1)$ and $f(1)$ equal 1. So two different values of x in the domain can give the same output value from the function. However, if we reverse the mapping, an input of 1 results in not one, but two distinct outputs, 1 and -1.

Recall that $\mathbb{R}$ is the set of all real numbers.

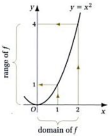
A one-to-many mapping.

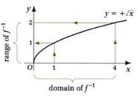




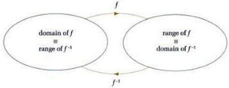
It is, however, possible to define an inverse function if we restrict the domain to values of x for which the mapping is one-to-one. For example, if $f: x \to x^2$ is defined for the restricted domain $\{x \in \mathbb{R}: 0 \leq x \leq 2\}$ then the inverse function $f^{-1}: x \to +\sqrt{x}$ exists.

The + sign indicates that only the positive square root is taken.





It is important to realise that the inverse function maps values in the range of the original function back onto the domain of the original function.



The graphical representation of a function can be used to determine whether it is a one-to-one mapping. If it is a one-to-one mapping then it has an inverse function.

Example 1

Use a graphical calculator, or alternative method, to sketch the graphs of the following functions:

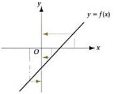
$f: x \to x - 3,\ x \in \mathbb{R}$

$g: x \to x^3,\ x \in \mathbb{R}$

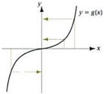
$h: x \to x^3 - 12x,\ x \in \mathbb{R}$.

Use the graphs to decide which of these functions have inverse functions.

Solution

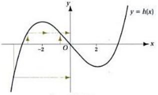


$f{:}x \to x - 3$



$g{:}x \to x^3$

Functions f and g are both one-to-one mappings. Therefore both have inverse functions.

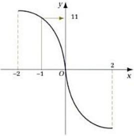


$h{:}x \to x^3 - 12x$

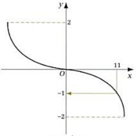
Notice that the graph of $y = h(x)$ has two **turning points**; at $x = -2$ and $x = 2$. It doubles back on itself between these two points. Function h is therefore not a one-to-one mapping and so, for the given domain, h does not have an inverse function.

A turning point is a maximum or minimu point on a graph.

If the domain of function h is restricted to a set of values of x for which the mapping is one-to-one, for example $\{x \in \mathbb{R}: -2 \leq x \leq 2\}$, then the inverse function h^{-1} exists. Since $h(-1) = 11$, it follows that, $h^{-1}(11) = -1$.



$y = h(x)$



$y = h^{-1}(x)$

The graph of an inverse function

If a function f maps values of x in its domain to values of y in its range then the inverse function f^{-1} maps values of y back to the corresponding values of x. It follows that for every point (x, y) on the graph of function f there is a corresponding point with coordinates (y, x) on the graph of the inverse function f^{-1}. Geometrically, this interchanging of x and y is equivalent to a reflection in the line $y = x$.

The graph of the inverse function $y = f^{-1}(x)$ can be sketched by simply reflecting the graph of $y = f(x)$ in the line $y = x$.

To carry out this reflection properly, it is important to use the same scale on both axes.

Example 2

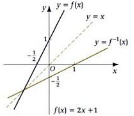
Sketch each of the following graphs, and draw in the line $y = x$. Reflect each of the graphs in $y = x$ to obtain the graph of the inverse function.

Remember to use the same scale for both axes.

a $f(x) = 2x + 1$
b $f(x) = 3 - x$
c $f(x) = x^3 + 1$

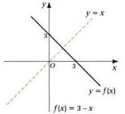
Solution

a



◀ $(0, 1)$ corresponds to $(1, 0)$, and $(-\frac{1}{2}, 0)$ corresponds to $(0, -\frac{1}{2})$

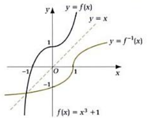
b



◀ $(3, 0)$ corresponds to $(0, 3)$, and $(0, 3)$ corresponds to $(3, 0)$

The line $y = 3 - x$ is symmetrical about the line $y = x$. This means that $f(x)$ is its own inverse; **self-inverse.** Therefore $f^{-1}(x) = 3 - x$.

c



◀ $(0, 1)$ corresponds to $(1, 0)$, and $(-1, 0)$ corresponds to $(0, -1)$

Graphical calculator support pack

Finding the inverse function

Two different methods can be used to find an algebraic expression for the inverse of a function:

- express the function as a 'flow diagram' or 'function machine', and then reverse the direction,
- express the function in the form $y = f(x)$, interchange x and y, and then rearrange the equation to make y the subject of the equation again.

Example 3

a Find the inverse function of $f(x) = 2x + 5$.

b The function $f: x \to \frac{2}{x}$ is defined for $\{x \in \mathbb{R}: x \neq 0\}$. Find its inverse function.

Solution

a *Method 1:* Express the function as a 'function machine'.

$$x \xrightarrow{\times 2} 2x \xrightarrow{+5} f(x) = 2x + 5$$

The inverse function is found by reversing each of the individual operations and the order in which they are carried out.

$$f^{-1}(x) = \frac{x-5}{2} \xleftarrow{\div 2} x - 5 \xleftarrow{-5} x$$ ◀ **Read from right to left.**

Therefore, the inverse function is $f^{-1}(x) = \frac{1}{2}(x - 5)$.

Try this approach the functions in Example 2. Notice this particular met will only work if x appears only once function.

Method 2: Expressing the function in the form $y = 2x + 5$, and interchanging x and y,

$$x = 2y + 5$$

Now make y the subject of the equation.

$$y = \tfrac{1}{2}(x - 5)$$

Interchanging x an the algebraic equiv of reflecting the function in the line $y = x$.

So, as before, the inverse function is $f^{-1}(x) = \frac{1}{2}(x - 5)$.
Use a graphical calculator to draw the lines $y = 2x + 5$, $y = \frac{1}{2}(x - 5)$ and $y = x$. Check that the graph of the inverse function is a reflection of the graph of the function in the line $y = x$. Remember to use the same scale on both axes.

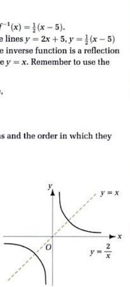
b The function machine for f looks like,

$$x \xrightarrow{\text{invert}} \frac{1}{x} \xrightarrow{\times 2} f(x) = \frac{2}{x}$$

Now reverse the individual operations and the order in which they are applied:

$$f^{-1}(x) = \frac{2}{x} \xleftarrow{\text{invert}} \frac{x}{2} \xleftarrow{\div 2} x$$

Therefore $f^{-1}: x \to \frac{2}{x}$, and function f is another example of a self-inverse. Draw the graph of $y = \frac{2}{x}$. What do you notice? The graph of $y = \frac{2}{x}$ is symmetrical about the line $y = x$.



Use a graphical calculator to draw the line $y = x$ and the graphs of functions of the form $f(x) = a - x$ and $f(x) = \frac{a}{x}$ for different values of the real number a. What happens? You should discover that all functions of the form $f(x) = a - x$ and $f(x) = \frac{a}{x}$, where a is a real number, are symmetrical about the line $y = x$. They are all self-inverses.

Example 4

The function $g(x) = \dfrac{x+1}{x-3}$ is defined for the domain $\{x \in \mathbb{R}: x \neq 3\}$. Find its inverse function, and state the value of x for which $g^{-1}(x)$ is not defined.

Solution

Writing the function in the form $y = g(x)$, and interchanging x and y gives $x = \dfrac{y+1}{y-3}$.

Rearrange this equation to make y the subject,

$$x(y-3) = y+1$$

$$\Rightarrow \quad xy - 3x = y + 1$$

$$\Rightarrow \quad xy - y = 3x + 1$$

$$\Rightarrow \quad y(x-1) = 3x + 1$$

$$y = \frac{3x+1}{x-1}$$

$$\text{So} \quad g^{-1}(x) = \frac{3x+1}{x-1}$$

Try putting $x = 1$ into the inverse function. What happens? Division by zero causes a problem, creating an answer that is undefined. So this inverse function is defined for all real values of x except $x = 1$.

Example 5

The quadratic function $f(x) = x^2 - 8x + 14$ is defined for the domain $\{x \in \mathbb{R}: x \geq 4\}$.

a By 'completing the square', express f in the form $(x+a)^2 + b$, where a and b are integers.

b Hence, or otherwise, find an expression for the inverse function f^{-1}.

c Sketch the graph of $y = f^{-1}(x)$, and state the domain and range of this inverse function.

d Solve the equation $f(x) = f^{-1}(x)$.

Solution

a $f(x) = x^2 - 8x + 14$

$= x^2 - 8x + 16 - 2$

$= (x - 4)^2 - 2$

$\Rightarrow \quad a = -4$ and $b = -2$.

Completing the square and spotting that $16 = 4^2$.

b Writing the function in the form $y = (x - 4)^2 - 2$, and interchanging x and y,

$x = (y - 4)^2 - 2$

Rearranging to make y the subject,

$(y - 4)^2 = x + 2$

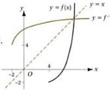
$\Rightarrow \quad y - 4 = \sqrt{x + 2}$

$\Rightarrow \quad y = 4 + \sqrt{x + 2}$

Take the positive square root.

Therefore, the inverse function is $f^{-1}(x) = 4 + \sqrt{x + 2}$.

c Reflecting the graph of $y = f(x)$ in the line $y = x$ gives the graph of $y = f^{-1}(x)$.
The domain of f^{-1} is $\{x \in \mathbb{R}: x \geq -2\}$.
The range of f^{-1} is $\{y \in \mathbb{R}: y \geq 4\}$.



d Equating $f(x) = x^2 - 8x + 14$ with $f^{-1}(x) = 4 + \sqrt{x + 2}$ gives

$x^2 - 8x + 14 = 4 + \sqrt{x + 2}$

Although progress can be made with this equation, it will involve cubic (x^3) and quartic (x^4) terms. Look at the graphs. Notice that the point at which $f(x) = f^{-1}(x)$ is also the point of intersection of the line $y = x$ and the curve $y = f(x)$, so solving the simultaneous equations $y = x$ and $y = f(x)$ may be an easier way to find the solution.

$x = x^2 - 8x + 14$

If $y = x$ and $y = f(x)$ then $x = f(x)$.

$\Rightarrow x^2 - 9x + 14 = 0$ ◀ **Collecting like terms.**

$\Rightarrow (x - 2)(x - 7) = 0$ ◀ **Factorising.**

$\Rightarrow x = 2$ or $x = 7$

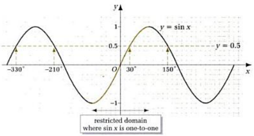
Since function f is only defined for $x \geq 4$, ignore the solution $x = 2$. Therefore, the only solution to the equation $f(x) = f^{-1}(x)$ is $x = 7$.

Sometimes it may be easier to solve the equation $f(x) = f^{-1}(x)$ by finding the intersection of $y = x$ and $y = f^{-1}(x)$. This is the same as solving $x = f^{-1}(x)$.

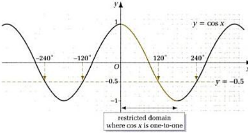
Inverse trigonometric functions

The sine, cosine and tangent functions are periodic. For the domain of real numbers they are many-to-one mappings, as illustrated.

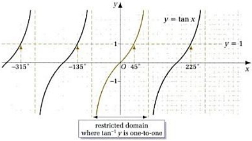
The coloured sections show the restricted domains where the functions are one-to-one mappings.



$\sin x = 0.5$ for $x = -330°, -210°, 30°, 150°$, and so on.



$\cos x = -0.5$ for $x = -240°, -120°, 120°, 240°$, and so on.



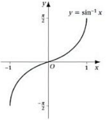
$\tan x = 1$ for $x = -315°, -135°, 45°, 225°$, and so on.

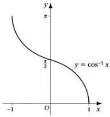
The inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ can each be defined for a restricted domain. These domains are the values of x for which the sine, cosine and tangent mappings are one-to-one. The restrictions are different for each of the three trigonometric functions.

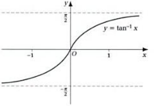
Another notation is arcsin, arccos and arctan.

For $f(x) = \sin x$ and $f(x) = \tan x$ to be one-to-one mappings, $-90^\circ \le x^\circ \le 90^\circ$, or $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ if x measured in radians. However, for $f(x) = \cos x$ to be a one-to-one mapping, $0^\circ \le x^\circ \le 180^\circ$, or $0 \le x \le \pi$ if x is measured in radians.

The graphs of $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} x$ can be sketched by reflecting the graphs of $y = \sin x$, $y = \cos x$ or $y = \tan x$ in the line $y = x$. Remember to use the same scale on both axes and to measure the angles in radians.





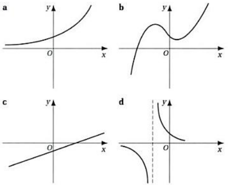


4.2 Inverse Functions
Exercise
Technique

1 For each of the following functions:

i state whether an inverse function exists, giving a reason for your answer

ii sketch the graph of the inverse function if it exists.



2 Find an expression for f^{-1} for each of the following functions:

a $f: x \to 3x - 8$

b $f: x \to \dfrac{4}{x+3},\ x \neq -3$

c $f: x \to 2x^2 + 7,\ x \geq 0$

d $f: x \to \frac{1}{2}\sqrt{x^2 + 5},\ x \geq 0$

e $f: x \to \dfrac{2}{x} + 3,\ x \neq 0$

f $f: x \to \dfrac{4}{x^2},\ x > 0$

2 a, b, c

3 The rational functions f, g and h are defined on the set of real numbers such that:

$$f(x) = \frac{3}{x-4}, \qquad g(x) = \frac{2x}{x+1}, \qquad h(x) = \frac{2x+3}{x-5}$$

a State the values of x for which each of these functions are undefined.

b Find an expression for each of the inverse functions, and state their domains.

4 The functions f and g are defined such that

$$f(x) = \frac{ax+2}{x-3}, \qquad g(x) = \frac{5x-1}{2x+b}$$

where a and b are constants.

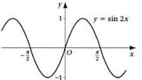
a Find expressions for $f^{-1}(x)$ and $g^{-1}(x)$.
b State the values of a and b for which functions f and g are self-inverses.

5 **a** Express $x^2 - 2x - 4$ in the form $(x+a)^2 + b$, where a and b are integers.
b Function $f(x) = x^2 - 2x - 4$ is defined for the domain $x \in \mathbb{R}: x \geq q$. Find the least value of q for which function f is a one-to-one mapping.
c State the range of f.
d Find an expression for the inverse function f^{-1}.
e Solve the equation $f(x) = f^{-1}(x)$.

6 The function f is defined for the domain $0 \leq x \leq \frac{\pi}{9}$ by $f(x) = \cos 3x$.

a Find an expression for f^{-1}. State the domain of f^{-1}.
b Calculate the value of $f^{-1}(\frac{\sqrt{3}}{2})$ in terms of π.

7



The diagram shows the graph of $y = f(x)$, where $f(x) = \sin 2x$.

a State the restricted domain for which function f is a one-to-one mapping.
b For this restricted domain:
  i find an expression for f^{-1}
  ii state the domain of f^{-1}
  iii sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on one set of axes, using the same scale on both axes.
c State the domain of the function $g(x) = \sin ax$, where a is a positive constant, if the inverse function g^{-1} exists.

4.3 Composite Functions

The output values of two or more functions can be combined using addition, subtraction, multiplication and division. If $f(x) = 3x$ and $g(x) = \sin x$, possible combinations include:

$$f(x) + g(x) = 3x + \sin x$$

$$f(x) - g(x) = 3x - \sin x$$

$$f(x) \times g(x) = 3x \sin x$$

$$f(x) \div g(x) = \frac{3x}{\sin x} \text{ (provided } \sin x \neq 0\text{)}$$

It is also possible to take the output values from one function and use them as the input values for another function. Using $f(x) = 3x$ and $g(x) = \sin x$, first evaluate function g for different values of x, and then input the values of $g(x)$ into function f. What happens?

$$x \xrightarrow{g} g(x) = \sin x \xrightarrow{f} f(g(x)) = 3\sin x$$

The resulting composite function (or 'function of a function') is $f(g(x)) = 3\sin x$, and is commonly referred to as $fg(x)$. Note that the g is written closest to the x because function g is the first stage of this composite function.

An alternative notation for the composite function $fg(x)$ is $f \circ g(x)$.

Now evaluate function f first and use the corresponding values of $f(x)$ as inputs into function g. What happens?

$$x \xrightarrow{f} f(x) = 3x \xrightarrow{g} g(f(x)) = \sin 3x$$

Notice that the composite function isn't the same. This time the output is $gf(x) = \sin 3x$.

Example 1

The functions f, g and h are defined by $f: x \to \cos x$, $g: x \to x^2$, $h: x \to 2x + 1$. Find expressions for the following composite functions:

a $gf(x)$	**b** $hg(x)$	**c** $gh(x)$
d $hf(x)$	**e** $hgf(x)$	**f** $fhg(x)$

Solution

a
$$\begin{aligned} gf(x) &= g(f(x)) \\ &= g(\cos x) \\ &= (\cos x)^2 \\ &= \cos^2 x \end{aligned}$$

Input $\cos x$ into function g.

Recall the conventional way of writing $(\cos x)^2$.

b $hg(x) = h(g(x))$
$= h(x^2)$
$= 2(x^2) + 1$
$= 2x^2 + 1$

x^2 replaces x in func
h.

c $gh(x) = g(h(x))$
$= g(2x + 1)$
$= (2x + 1)^2$

d $hf(x) = h(f(x))$
$= h(\cos x)$
$= 2(\cos x) + 1$
$= 2\cos x + 1$

$\cos x$ replaces x in function h.

e $hgf(x) = h(g(f(x)))$
$= h(g(\cos x))$
$= h(\cos^2 x)$
$= 2(\cos^2 x) + 1$
$= 2\cos^2 x + 1$

Input composite function $gf(x)$ into function h.

f $fhg(x) = f(hg(x))$
$= f(2x^2 + 1)$
$= \cos(2x^2 + 1)$

From **b**, $hg(x) = 2x^2$

Example 2

Each of the following functions are composite functions of the form $gf(x)$. In each case find the component functions f and g.

a $p(x) = \dfrac{1}{x+2}$ **b** $q(x) = 8x^3$ **c** $r(x) = 5 - x$

Solution

Use flow diagrams to decompose each of these functions.

a $p(x)\colon x \xrightarrow{+2} x + 2 \xrightarrow{\text{invert}} \dfrac{1}{x+2}$

Therefore $p(x) \equiv gf(x)$, where $f(x) = x + 2$ and $g(x) = \frac{1}{x}$.

b $q(x)\colon x \xrightarrow{\text{cube}} x^3 \xrightarrow{\times 8} 8x^3$

Therefore $q(x) \equiv gf(x)$, where $f(x) = x^3$ and $g(x) = 8x$.
There is an alternative way of expressing function q as the composite of two functions.

$q(x)\colon x \xrightarrow{\times 2} 2x \xrightarrow{\text{cube}} 8x^3$

Then $q(x) \equiv gf(x)$, where $f(x) = 2x$ and $g(x) = x^3$.

c $r(x): x \xrightarrow{\times(-1)} -x \xrightarrow{+5} -x+5 = 5-x$

Therefore $r(x) \equiv gf(x)$, where $f(x) = -x$ and $g(x) = x+5$.
Again, there is an alternative.

$r(x): x \xrightarrow{-5} x-5 \xrightarrow{\times(-1)} -(x-5) = 5-x$

Then $r(x) \equiv gf(x)$, where $f(x) = x-5$ and $g(x) = -x$.

Instead of feeding the output from one particular function into a different function, we could input back into the original function itself. Using $f(x) = 3x$ again, the composite function

$$ff(x) = f(f(x)) = f(3x) = 9x$$

Note that $ff(x)$ is often referred to as $f^2(x)$, indicating that the function is to be carried out twice (*and not that the output from the function is to be squared*). Similarly, for this particular function,

$$f^3(x) \equiv fff(x) = f(f(f(x))) = 3(3(3x)) = 3(9x) = 27x$$

Example 3

The functions f, g and h are defined by

$$f(x) = 2-x, \qquad g(x) = \frac{3}{x+1} \quad (x \neq -1), \qquad \text{and} \qquad h(x) = 2x-1$$

a Show that $f^2(x) = x$.
b Find an expression for $g^2(x)$, and state for which two values of x it is undefined.
c Solve the equation $h^3(x) = x$.

Solution

a
$$\begin{aligned} f^2(x) &= f(f(x)) \\ &= f(2-x) \\ &= 2-(2-x) = x \end{aligned}$$

b
$$\begin{aligned} g^2(x) &= g(g(x)) \\ &= g\left(\frac{3}{x+1}\right) \\ &= \frac{3}{\frac{3}{x+1}+1} \\ &= \frac{3}{\frac{x+4}{x+1}} = \frac{3(x+1)}{x+4} \end{aligned}$$

Making $x+1$ the common denominator of the fraction in the denominator.

Since $g(x)$ is undefined for $x = -1$, it is not possible to evaluate $g^2(x)$ when $x = -1$. You can see that $g^2(x)$ is also undefined for $x = -4$.

c $h^2(x) = h(h(x))$

$$\begin{aligned} &= h(2x-1) \\ &= 2(2x-1) - 1 \\ &= 4x - 3 \end{aligned}$$

Now $h^3(x) = h(h^2(x))$

$$\begin{aligned} &= h(4x-3) \\ &= 2(4x-3) - 1 \\ &= 8x - 7 \end{aligned}$$

So when $h^3(x) = x$,

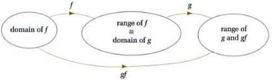
$$8x - 7 = x$$

$$\Rightarrow \quad 7x = 7$$

$$\Rightarrow \quad x = 1$$

Domain of a composite function

The relationships between the domains and ranges of functions f and g, and the range of composite function gf are illustrated here.



How is the domain of a composite function found? First, establish which values of x in the domain of function f produce values of $f(x)$ that lie in the domain of function g. These can then be fed into $g(x)$. However, depending upon the nature of functions f and g, and the values of x in the domains for which they are defined, the composite function gf may have a restricted domain.

Example 4

The functions f and g are defined by

$$f(x) = 1 + 2x \text{ for } \{x \in \mathbb{R}: x \geq 0\}$$

$$g(x) = \frac{1}{x-1} \text{ for } \{x \in \mathbb{R}: x > 1\}$$

a State the range of f and g for the given domains.

b Find an expression for $gf(x)$, and determine its domain.

c Find an expression for $fg(x)$, and determine its domain.

Solution

a For the given domain, the least value of $f(x)$ is 1. This is when $x = 0$. Therefore the range of $f(x)$ is $\{y \in \mathbb{R}: y \geq 1\}$.

What about $g(x)$? As x increases, $x - 1$ also increases. This means that as x increases $\frac{1}{x-1}$ decreases. As x becomes very large, $g(x)$ will tend towards zero. Conversely, for values of x close to 1, the denominator is very small and $g(x)$ becomes very large. This means that the range of $g(x)$ is $\{y \in \mathbb{R}: y > 0\}$.

b $gf(x) = g(1 + 2x)$

$$= \frac{1}{(1 + 2x) - 1} = \frac{1}{2x}$$

The domain of the composite function gf is the set of values of x in the domain of function f that give values of $f(x)$ that are in the domain of function g.

What about the domain of g? The domain of function g is $\{x \in \mathbb{R}: x > 1\}$, so we require the values of x in the domain of f for which $f(x) > 1$. Are there values in the domain of f that do not satisfy this? The only value of x in the domain of f that does not satisfy this condition is $x = 0$. This means that the domain of the composite function $gf(x)$ is $\{x \in \mathbb{R}: x > 0\}$.

c $fg(x) = f\left(\frac{1}{x-1}\right)$

$$= 1 + 2\left(\frac{1}{x-1}\right)$$

$$= \frac{x - 1 + 2}{x - 1} = \frac{x + 1}{x - 1}$$

The domain of the composite function fg is the set of values of x in the domain of function g which give values of $g(x)$ that are in the domain of function f.

What about the domain of f? The domain of function f is $\{x \in \mathbb{R}: x > 0\}$. This means that we require $g(x) \geq 0$. Are there any values in the domain of g that do not satisfy this? All values of x in the domain of g satisfy the condition $g(x) \geq 0$. So the domain of the composite function $fg(x)$ is $\{x \in \mathbb{R}: x > 1\}$; that is, the entire domain of g.

The inverse of a composite function

The flow diagram shows how the composite function fg is obtained by putting values of x into g, and then putting values of $g(x)$ into f.

$$x \xrightarrow{g} g(x) \xrightarrow{f} fg(x)$$

The inverse of this composite function is $(fg)^{-1}$. This is obtained by reversing each of the individual component functions and the order in which they are applied.

$$g^{-1}f^{-1} \xleftarrow{g^{-1}} f^{-1}(x) \xleftarrow{f^{-1}} x$$

Read from right to le

So values of x are put into f^{-1}, and then values of $f^{-1}(x)$ are put into g^{-1}. This gives:

$$(fg)^{-1}(x) \equiv g^{-1}f^{-1}(x)$$

Example 5

The functions f and g defined for $x \in \mathbb{R}$ by $f: x \to x^3$ and $g: x \to 2x + 1$. Find:

a f^{-1} and g^{-1} **b** fg and $(fg)^{-1}$ **c** gf and $(gf)^{-1}$

Solution

a $f^{-1}: x \to \sqrt[3]{x}$

$g^{-1}: x \to \dfrac{x-1}{2}$

Check these by substituting values f the function into the inverse function.

b $fg(x) = f(2x + 1) = (2x + 1)^3$

$$\begin{aligned}(fg)^{-1}(x) &= g^{-1}f^{-1}(x)\\ &= g^{-1}(\sqrt[3]{x})\\ &= \frac{\sqrt[3]{x}-1}{2}\end{aligned}$$

Check this result by substituting values f

c $gf(x) = g(x^3) = 2x^3 + 1$

$$\begin{aligned}(gf)^{-1}(x) &= f^{-1}g^{-1}(x)\\ &= f^{-1}\left(\frac{x-1}{2}\right)\\ &= \sqrt[3]{\frac{x-1}{2}}\end{aligned}$$

Check this result by substituting values f

Note that having found the composite function, the inverse can also be found by writing it in the form $y = fg(x)$ (or $y = gf(x)$), interchanging the x and y, and then making y the subject of the equation.

Applying this technique to Example 5**b**, writing fg as $y = (2x + 1)^3$ and then interchanging x and y gives $x = (2y + 1)^3$. Making y the subject of this equation, $y = \frac{1}{2}(\sqrt[3]{x} - 1)$. So $(fg)^{-1}(x) = \frac{1}{2}(\sqrt[3]{x} - 1)$.

Applying this technique to Example 5**c**, writing gf as $y = 2x^3 + 1$ and then interchanging x and y gives $x = 2y^3 + 1$. Making y the subject of this equation, $y = \sqrt[3]{\frac{1}{2}(x-1)}$. So $(gf)^{-1}(x) = \sqrt[3]{\frac{1}{2}(x-1)}$, as before.

4.3 Composite Functions

Exercise

Technique

1 The functions f, g and h are defined by $f(x) = x^2$, $g(x) = \frac{3}{x}$ and $h(x) = 2 - x$. Find an expression for each of the following composite functions in terms of x.

a fg	**b** gf	**c** fh	**d** hf	**e** gh
f hg	**g** g^2	**h** h^2	**i** ghf	**j** hgf

2 For each of the following, express function p as a composite of the functions $f\colon x \to 5x$, $g\colon x \to x + 3$ and $h\colon x \to \sin x$:

a $p\colon x \to 5x + 3$

b $p\colon x \to \sin(x + 3)$

c $p\colon x \to \sin x + 3$

d $p\colon x \to 5\sin x + 15$

e $p\colon x \to x + 6$

f $p\colon x \to \sin(5x + 15)$.

3 Given that $f(x) = x - 3$, $g(x) = 10x$ and $h(x) = \frac{1}{x}$ $(x \neq 0)$:

a find an expression for $fgh(x)$

b solve the equation $fgh(x) = x$

4 Functions f and g are defined by $f(x) = 3x + 5$ and $g(x) = \dfrac{x - 5}{3}$.

a Find expressions for $f^2(x)$ and $g^2(x)$.

b Show that $fg(x) = x$ and $gf(x) = x$.

c Comment on the significance of your results in **b**.

5 The functions f and g are defined by $f\colon x \to 4 - x^2$ for $\{x \in \mathbb{R}\}$ and $g\colon x \to \sqrt{x}$ for $\{x \in \mathbb{R}, x \geq 0\}$.

a State the ranges of f and g for the given domains.

b Find an expression for fg, and determine its domain.

c Find an expression for gf, and determine its domain.

6 The functions f and g are defined by $f\colon x \to \frac{1}{4}x^2$ for $\{x \in \mathbb{R}\}$ and $g\colon x \to \dfrac{2}{x - 1}$ for $\{x \in \mathbb{R}, x \neq 1\}$. Find expressions for the following, in each case stating any values of x for which the composite functions are undefined:

a $fg(x)$ **b** $gf(x)$ **c** $f^2(x)$ **d** $g^2(x)$

7 Given that $f(x) = x^3$, $g(x) = 1 - 3x$ and $h(x) = \frac{2}{x}$ $(x \neq 0)$, find expressions for:

a f^{-1}, g^{-1} and h^{-1}

b fg, $(fg)^{-1}$ and $g^{-1}f^{-1}$

c hg, $(hg)^{-1}$ and $g^{-1}h^{-1}$

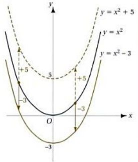
d fh, $(fh)^{-1}$ and $h^{-1}f^{-1}$

4.4 Transformations of Graphs and Functions

Translations

Graphs of the form $y = f(x) + a$

Using a graphical calculator draw the graphs of $y = x^2$, $y = x^2 + 5$ and $y = x^2 - 3$ on the same axes. How are the graphs of $y = x^2 + 5$ and $y = x^2 - 3$ related to the graph of $y = x^2$? Notice that the shape of the curve is the same. The curve has been **translated**, or moved.



Graphic calculat support pack

The three curves all have the same shape. Although they appear to be getting closer together, their vertical separation at every value of x is constant; the graph of $y = x^2 + 5$ is 5 units above the graph $y = x^2$, while the graph of $y = x^2 - 3$ is 3 units below $y = x^2$. In fact the curve $y = x^2 + 5$ is obtained by simply **translating** the curve $y = x^2$ by $+5$ units parallel to the y-axis and the curve $y = x^2 - 3$ is obtained by translating the curve $y = x^2$ by -3 units parallel to the y-axis. This can be generalised as follows.

The graph of $y = f(x) + a$ is obtained by translating the graph of $y = f(x)$ through a units parallel to the y-axis.

Graphs of the form $y = f(x + a)$

Using a graphical calculator, draw the graphs of $y = x^2$, $y = (x + 3)^2$ and $y = (x - 4)^2$ on the same axes. How are the graphs of $y = (x + 3)^2$ and $y = (x - 4)^2$ related to the graph of $y = x^2$?

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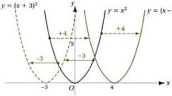


The graphs of $y = x^2$, $y = (x + 3)^2$ and $y = (x - 4)^2$ all have the same shape. The curve $y = (x + 3)^2$ is obtained by translating each point on the curve $y = x^2$ by -3 units parallel to the x-axis. The graphs of $y = x^2$ and $y = (x + 3)^2$ have the same value of y when the x-coordinate on the curve $y = (x + 3)^2$ is 3 units smaller than that on the curve $y = x^2$. The curve $y = (x - 4)^2$ is obtained by translating the curve $y = x^2$ by $+4$ units parallel to the x-axis. These results can be generalised as follows.

The graph of $y = f(x + a)$ is obtained by translating the graph of $y = f(x)$ through $-a$ units parallel to the x-axis.

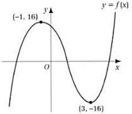
Example 1

The diagram shows the graph of $y = f(x)$ for $f(x) = x^3 - 3x^2 - 9x + 11$. Functions $g(x)$ and $h(x)$ are related to $f(x)$ with $g(x) = f(x - 1)$ and $h(x) = f(x + 2) - 3$.

a Explain how the graphs of $y = g(x)$ and $y = h(x)$ can be obtained from the graph of $y = f(x)$.

b Sketch both of these graphs, indicating the coordinates of their turning points.

c Show that $g(x) = x^3 - 6x^2 + 16$ and $h(x) = x^3 + 3x^2 - 9x - 14$.

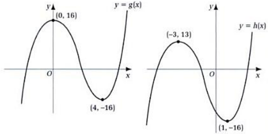


Solution

a The graph of $y = g(x) = f(x - 1)$ is obtained by translating the $y = f(x)$ graph by $+1$ unit parallel to the x-axis.

The graph of $y = h(x) = f(x + 2) - 3$ is obtained by translating the $y = f(x)$ graph by -2 units parallel to the x-axis, and -3 units parallel to the y-axis.

b



Notice how the coordinates of the turning points have been translated.

c $g(x) = f(x-1)$

$$= (x-1)^3 - 3(x-1)^2 - 9(x-1) + 11$$
$$= (x^3 - 3x^2 + 3x - 1) - (3x^2 - 6x + 3) - (9x - 9) + 11$$
$$= x^3 - 6x^2 + 16, \text{ as required}$$

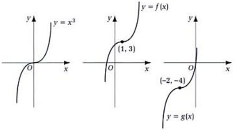
Expand the brackets, collect like terms and simplify.

$h(x) = f(x+2) - 3$

$$= (x+2)^3 - 3(x+2)^2 - 9(x+2) + 11 - 3$$
$$= (x^3 + 6x^2 + 12x + 8) - (3x^2 + 12x + 12) - (9x + 18) + 8$$
$$= x^3 + 3x^2 - 9x - 14, \text{ as required}$$

Example 2

The diagram shows the graph of $y = x^3$. The graphs of $y = f(x)$ and $y = g(x)$ have been obtained by translating the curve $y = x^3$.



a Describe the transformation by which the graph of $y = x^3$ is mapped to the graphs of $y = f(x)$ and $y = g(x)$.

b Find expressions for $f(x)$ and $g(x)$.

Solution

a The graph of $y = f(x)$ is obtained by translating the curve $y = x^3$ by $+1$ unit parallel to the x-axis, and $+3$ units parallel to the y-axis.
The graph of $y = g(x)$ is obtained by translating the curve $y = x^3$ by -2 units parallel to the x-axis, and -4 units parallel to the y-axis.

b $f(x) = (x-1)^3 + 3$

The x-axis translation is the -1, and the y-axis translation is the $+3$.

Expanding the brackets and collecting like terms,

$f(x) = x^3 - 3x^2 + 3x + 2$

$g(x) = (x+2)^3 - 4$

The x-axis translation is the $+2$, and the y-axis translation is the -4.

Expanding the brackets and collecting like terms,

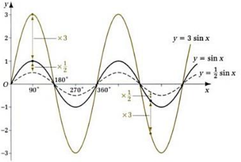
$g(x) = x^3 + 6x^2 + 12x + 4$

Stretches

Graphs of the form $y = af(x)$

Using a graphical calculator, draw the graphs of $y = \sin x$, $y = 3\sin x$ and $y = \frac{1}{2}\sin x$ on the same axes. How are the graphs of $y = 3\sin x$ and $y = \frac{1}{2}\sin x$ related to the graph of $y = \sin x$?

Graphical calculator support pack



Look at the graphs of $y = \sin x$ and $y = 3\sin x$. Both curves repeat themselves with a period of 360°. This is a characteristic of the sine function. However, the curve $y = 3\sin x$ has an amplitude of 3. This is because its maximum and minimum values are $+3$ and -3 respectively. In fact, the curve $y = 3\sin x$ can be drawn by simply stretching the curve $y = \sin x$ parallel to the y-axis by a factor of 3 (**scale factor** 3).

Amplitude is the greatest height reached by the wave from the x-axis.

The graph $y = \frac{1}{2}\sin x$ also has a period of 360°. Notice that its amplitude is 0.5. It can be obtained by 'stretching' the curve $y = \sin x$ by a scale factor of $\frac{1}{2}$ parallel to the y-axis. Notice also that, a stretch factor between 0 and 1 actually has the effect of compressing the curve (that is making it closer to the x-axis).

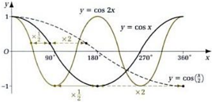
This can be generalised as follows:

The graph of $y = af(x)$ is obtained from the graph of $y = f(x)$ by a one-way stretch, of scale factor a, parallel to the y-axis.

Graphs of the form $y = f(ax)$

Using a graphical calculator, draw the graphs of $y = \cos x$, $y = \cos 2x$ and $y = \cos(\frac{x}{2})$ on the same axes. How are the graphs of $y = \cos 2x$ and $y = \cos(\frac{x}{2})$ related to the graph of $y = \cos x$? Explain the transformation required to obtain the graph of $y = f(ax)$ from the graph of $y = f(x)$.

Graphic calculator support pack



The graph of $y = \cos x$ repeats itself every 360°. The curve $y = \cos 2x$ has a period of only 180°. It completes two full cycles every 360°. In fact, the curve $y = \cos 2x$ has been obtained by 'stretching' the graph of $y = \cos x$ parallel to the x-axis by a *scale factor* of $\frac{1}{2}$. This has the effect of compressing each cycle into half the space. Notice that this is a one-way stretch and that the amplitude of the cosine curve is not altered.

The graph of $y = \cos(\frac{x}{2})$ completes half of a full cycle within 360°. It has a period of 720°. It is obtained from the graph of $y = \cos x$ by a one-way stretch of scale factor 2 parallel to the x-axis.

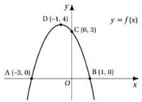
Stretches like this can be summarised as:

The graph of $y = f(ax)$ is obtained from the graph of $y = f(x)$ by a one-way stretch, scale factor $\frac{1}{a}$, parallel to the x-axis.

Example 3

The diagram shows the graph of $y = f(x)$, where $f(x) = 3 - 2x - x^2$. This graph is mapped to $y = g(x)$ by a stretch of factor 2 parallel to the y-axis.

The graph of $y = g(x)$ is then itself mapped to $y = h(x)$ by a stretch of factor 3 parallel to the x-axis.

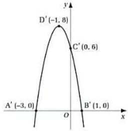


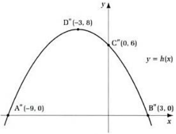
a Sketch the graphs of $y = g(x)$ and $y = h(x)$. Indicate clearly the coordinates of the images of points A, B, C and D.

b Find expressions for $g(x)$ and $h(x)$ in terms of x.

Solution

a





b The graph of $y = g(x)$ is obtained by stretching the graph of $y = f(x)$ by a factor of 2 parallel to the y-axis. So

$$\begin{aligned} g(x) &= 2f(x) \\ &= 2(3 - 2x - x^2) \\ &= 6 - 4x - 2x^2 \end{aligned}$$

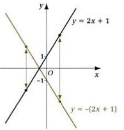
The graph of $y = h(x)$ is obtained by stretching the graph of $y = g(x)$ by a factor of 3 parallel to the x-axis. So

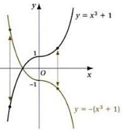
$$\begin{aligned} h(x) &= g(\tfrac{1}{3}x) \\ &= 6 - \frac{4x}{3} - 2\left(\frac{x}{3}\right)^2 \\ &= 6 - \frac{4x}{3} - \frac{2x^2}{9} \end{aligned}$$

Reflections

Graphs of the form $y = -f(x)$

Using a graphical calculator, draw the graphs of $y = 2x + 1$ and $y = -(2x + 1)$. How are these two graphs related to each other?
Now draw the graphs of $y = x^3 + 1$ and $y = -(x^3 + 1)$. Are these two graphs related in the same way?





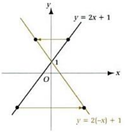
The graphs of $y = 2x + 1$ and $y = -(2x + 1)$ are shown. The latter has been obtained by reflecting the line $y = 2x + 1$ in the x-axis. Positive values of $y = 2x + 1$ become negative values of $y = -(2x + 1)$.
The curve $y = -(x^3 + 1)$ is similarly a reflection of the curve $y = x^3 + 1$ in the x-axis.

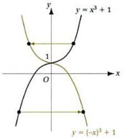
This can be generalised as follows:

The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

Graphs of the form $y = f(-x)$

Use a graphical calculator to draw the graphs of $y = 2x + 1$ and $y = 2(-x) + 1$ on the same axes. Now draw the graphs of $y = x^3 + 1$ and $y = (-x)^3 + 1$. How are these curves related to one another?





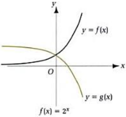
The graphs of $y = f(x)$ and $y = f(-x)$ for $f(x) = 2x + 1$ and $f(x) = x^3 + 1$ respectively are shown. In both examples, the graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the y-axis.

This transformation can be summarised as:

The graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the y-axis.

Example 4

The graph of $y = f(x)$ is mapped to the graph of $y = g(x)$ by a reflection in one axis, and a translation. Describe the transformations that have occurred. Clearly state the order in which they were carried out. Find an expression for $g(x)$.



Solution

Notice that there are two ways in which the graph of $y = 2^x$ can be mapped to $y = g(x)$:

- a reflection in the x-axis, followed by a translation of $+2$ units parallel to the y-axis;
- a translation of -2 units parallel to the y-axis, followed by a reflection in the x-axis.

Using the first of these, an expression for $g(x)$ can be found. Therefore $g(x) = -2^x + 2y$

$$y = 2^x$$
$$\downarrow \text{ reflection in } x\text{-axis}$$
$$= -2^x$$
$$\downarrow \text{ translation of } +2 \text{ units parallel to } y\text{-axis}$$
$$y = -2^x + 2$$

The second combination of transformations gives the same expression.

$$y = 2^x$$
$$\downarrow \text{ translation of } -2 \text{ units parallel to } y\text{-axis}$$
$$y = 2^x - 2$$
$$\downarrow \text{ reflection in } x\text{-axis}$$
$$y = -(2^x - 2) = -2^x + 2$$

The general quadratic curve

All quadratic expressions can be written in the form $c[(x + a)^2 + b]$, where a, b and c are constants. Why is this rearrangement useful? The graphs of $y = x^2$ and $y = c[(x + a)^2 + b]$ can be compared. The values of a, b and c then give an indication of the transformations used.

See Applications and Activities, Chapter 1.

Example 5

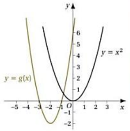
Express the function $g(x) = 2x^2 + 8x + 6$ in the form $c[(x + a)^2 + b]$, where a, b and c are constants. Describe the transformations by which the curve $y = x^2$ is mapped to the graph of this function. Sketch the graph of $y = g(x)$.

Solution

$$\begin{aligned} g(x) &= 2x^2 + 8x + 6 \\ &= 2(x^2 + 4x + 3) \\ &= 2(x^2 + 4x + 4 - 1) \\ &= 2[(x + 2)^2 - 1] \end{aligned}$$

Complete the square.

The graph of $y = x^2$ is mapped to the graph of $y = g(x)$ by translations of -2 units parallel to the x-axis and -1 unit parallel to the y-axis, followed by a stretch of factor 2 parallel to the y-axis.



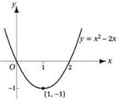
The order in which th translations are appli does not matter, although they *must* b applied before the stretch.

4.4 Transformations of Graphs and Functions

Exercise

Technique

1 The diagram shows the graph of $y = f(x)$, where $f(x) = x^2 - 2x$.



For each of the following transformations of this function:

i describe the transformation(s) geometrically;
ii write down an expression for the new function;
iii sketch the graph of the transformed function, indicating the coordinates of its intersection with the axes and the turning point:

a $y = f(x) + 2$ **b** $y = f(x - 3)$ **c** $y = f(2x)$
d $y = -3f(x)$ **e** $y = f(-\frac{1}{3}x)$ **f** $y = f(x + 1) + 4$

2 The function f is defined by $f(x) = x^2 + 6x + 8$.

a Express f in the form $f(x) = (x + a)^2 + b$, where a and b are integers.
b Describe the transformation by which the graph of $y = x^2$ is mapped to the graph of $y = f(x)$.

2

3 The function g is defined by $g(x) = 2x^2 - 12x + 19$.

a Express g in the form $g(x) = 2(x + a)^2 + b$, where a and b are integers.
b Describe the transformations by which the graph of $y = x^2$ is mapped to the graph of $y = g(x)$. Clearly state the order in which they must be applied.

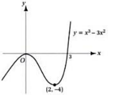
4 Each of the following quadratic curves is a transformation of the curve $y = x^2$. In each case, state the transformation(s) that have occurred. State also the order in which they must be applied.

a $y = x^2 + 3$
b $y = 4x^2$
c $y = (x - 2)^2$
d $y = x^2 + 2x + 5$
e $y = \frac{1}{2}x^2 + 1$
f $y = x^2 - 6x + 10$
g $y = x^2 + 4x$
h $y = 2x^2 - 4x + 3$

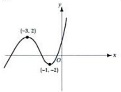
5 The diagram shows the graph of $y = x^3 - 3x^2$.

i Describe the transformations that map this graph to each of the following graphs. State the order in which they must be applied.

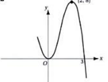
ii Use these transformations to write down the equation of each graph.



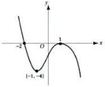
a



b



c



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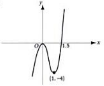


6 **i** Describe the transformations by which the graph of $y = \sin x$ is mapped to each of the curves below.

ii Make a sketch of the graph of each of the transformed functions for $-360° \le x \le 360°$.

a $y = \sin(x + 90°)$ **b** $y = \sin(\frac{x}{2})$

c $y = -2\sin x$ **d** $y = 3 + \sin x$

7 The function $f: x \to \frac{1}{x}, x \neq 0$ is transformed to give the function g. For each of the following transformations:

i find an expression for g;

ii state the value of x for which function g is undefined:

a a stretch of factor 3 parallel to the y-axis

b a translation of $+2$ units parallel to the x-axis

c a stretch of factor $\frac{1}{2}$ parallel to the x-axis, followed by a translation of -4 units parallel to the y-axis

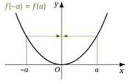
d a translation of -5 units parallel to the x-axis, followed by a stretch of factor 12 parallel to the y-axis.

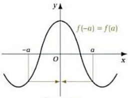
4.5 Even, Odd and Periodic Functions

Several of the functions considered so far have graphs that are symmetrical in some way. We can categorise these functions as even, odd or periodic.

Even functions

The graph of an even function has the y-axis as a line of symmetry. Examples of even functions include $f(x) = x^2$ and $f(x) = \cos x$.

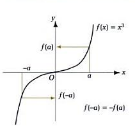


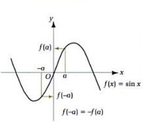


The reflection of the graph of $y = f(x)$ in the y-axis gives the graph of $y = f(-x)$. What does this suggest to you? Even functions satisfy the condition $f(-x) = f(x)$. This condition can be used to check whether or not a function is even.

Odd functions

The graphs of odd functions have rotational symmetry about the origin. Examples of odd functions include $f(x) = x^3$ and $f(x) = \sin x$.



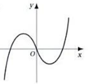


Odd functions satisfy the condition $f(-x) = -f(x)$. This condition can be used to check whether or not a function is odd.

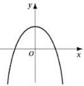
Example 1

State whether the following are graphs of even functions, odd functions, or neither:

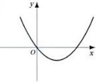
a



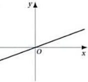
b



c



d



Solution

a The graph has rotational symmetry about the origin. This is the graph of an odd function.

b The graph is symmetrical about the y-axis. This is the graph of an even function.

c This graph has no rotational symmetrical about the origin and no reflective symmetry about the y-axis. The function is neither even nor odd.

d This is a graph of an odd function because it has rotational symmetry about the origin.

Example 2

The functions f, g and h are defined by $f(x) = x^3 - 2x$, $g(x) = 1 + x^2 - 5x^4$ and $h(x) = 3x^2 + 2x$. Determine which of these functions are even or odd.

Solution

Check whether or not a function is even or odd by substituting $-x$ into the function in place of x.

$$f(-x) = (-x)^3 - 2(-x)$$
$$= -x^3 + 2x$$
$$= -(x^3 - 2x)$$

So $f(-x) = -f(x)$

Writing this in terms the original function.

Therefore $f(x) = x^3 - 2x$ is an odd function.

$$g(-x) = 1 + (-x)^2 - 5(-x)^4$$
$$= 1 + x^2 - 5x^4$$

So $g(-x) = g(x)$

Therefore $g(x) = 1 + x^2 - 5x^4$ is an even function.

$$h(-x) = 3(-x)^2 + 2(-x)$$
$$= 3x^2 - 2x$$

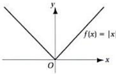
Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, function h is neither even nor odd.

The modulus function

The modulus function $f(x) = |x|$ reflects negative values into their positive equivalents. It is defined by:

$f(x) = x$, for $x \geq 0$ and $f(x) = -x$, for $x < 0$

Since its graph is symmetrical about the y-axis, the modulus function $f(x) = |x|$ is even.



To obtain the graph of $y = |f(x)|$, first sketch the graph of $y = f(x)$. The parts of this graph for which y is positive are kept: the parts for which y is negative are reflected in the x-axis.

Example 3

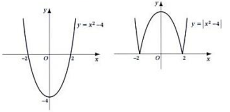
Sketch the graphs of:

a $y = |x^2 - 4|$ **b** $y = |\sin x|$

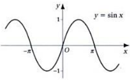
Graphical calculator support pack

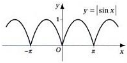
Solution

a First sketch the graph of $y = x^2 - 4$. Now reflect in the x-axis those parts of this graph that lie below the x-axis.



b Sketch the graph of $y = \sin x$ and then reflect in the x-axis those parts that lie below the x-axis.





The graphs of functions related to the modulus function can be sketched by applying the appropriate transformations to the graph of $y = |x|$.

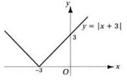
Example 4

Sketch the graphs of:

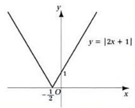
a $y = |x + 3|$ **b** $y = |2x + 1|$.

Solution

a The graph of $y = |x + 3|$ is obtained by translating the graph of $y = |x|$ by -3 units parallel to the x-axis.



b The graph of $y = |2x + 1|$ is obtained by stretching the graph of $y = |x|$ by a factor of $\frac{1}{2}$ parallel to the x-axis, and a translation of $-\frac{1}{2}$ parallel to the x-axis.



To obtain the graph of $y = f(|x|)$, first sketch the graph of $y = f(x)$ for positive values of x. For negative values of x, the value of $y = f(|x|)$ is found by substituting the equivalent positive value of x into the function. This simply reflects the graph of $y = f(x)$ for positive values of x in the y-axis.

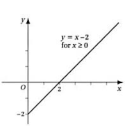
Example 5

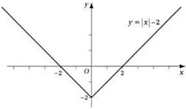
Sketch the graphs of:

a $y = |x| - 2$ **b** $y = \sin |x|$.

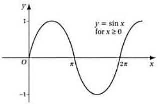
Solution

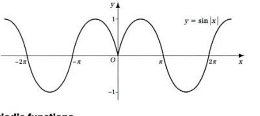
a The graph of $y = |x| - 2$ is obtained by first drawing the graph of $y = x - 2$ for positive values of x, and then reflecting this in the y-axis.





b The graph of $y = \sin|x|$ is obtained by first drawing the graph of $y = \sin x$ for positive values of x, and then reflecting this in the y-axis.





Periodic functions

Periodic functions have graphs that regularly repeat themselves. Examples of periodic functions include the sine, cosine and tangent functions. The graphs of $y = \sin x$ and $y = \cos x$ repeat themselves every 360° or 2π radians. This value is referred to as the **period** of the function. The graph of $y = \tan x$ has a period of only 180° or π radians.

In general, periodic functions $f(x)$ are such that for some constant k, $f(x \pm k) \equiv f(x)$, for all values of x, where k is the period of the function.

k is the smallest distance over which t function repeats itsel

Writing the trigonometric functions in this form

$$\sin(x \pm 360^\circ) \equiv \sin x \qquad \cos(x \pm 360^\circ) \equiv \cos x \qquad \tan(x \pm 180^\circ) \equiv \tan x$$

Once the behaviour over one period is known, the graph of a periodic function can be drawn.

Example 6

The function $f(x)$ is periodic with a period of 4 units. It is defined by

$$f(x) = \tfrac{1}{2}x^3, \qquad 0 \le x \le 2$$

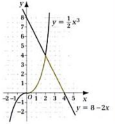
$$f(x) = 8 - 2x, \qquad 2 \le x \le 4$$

a Sketch the graph of $y = f(x)$ for $0 \le x \le 4$. Use its periodic behaviour to extend the graph to $-6 \le x \le 10$.

b Determine the values of $f(5)$ and $f(-1.5)$.

Solution

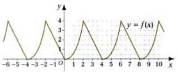
a Between 0 and 2, the graph is cubic in nature. Between 2 and 4 it is linear with a gradient of -2.



The coloured line sh the shape of the periodic function $f(x$ for $0 \le x \le 4$.

The function $f(x)$ repeats itself with a period of 4. Now sketch its graph for $-6 \le x \le 10$.

The function repeats itself every 4 units.



The function is cubic when $x = 1$.

The function is linea when $x = 2.5$.

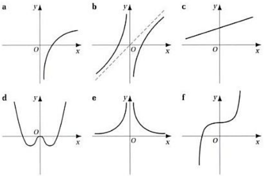
b
$$f(5) \equiv f(1) = \tfrac{1}{2}(1)^3 = \tfrac{1}{2} \qquad f(-1.5) \equiv f(2.5) = 8 - 2(2.5) = 3$$

4.5 Even, Odd and Periodic Functions

Exercise

Technique

1 Which of the following graphs represents the graph of an even or an odd function? 1



2 Which of the following functions are even, odd or neither even nor odd?

a $f(x) = x - \frac{1}{x}$

b $f(x) = 2x + |x|$

c $f(x) = \sin^2 x$

d $f(x) = (x+1)^2$

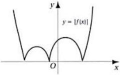
e $f(x) = (x+1)^3$

f $f(x) = \dfrac{1}{2x+1}$

g $f(x) = \dfrac{x}{2x+1}$

h $f(x) = (x^2+4)^3$

3 The diagram shows the graph of $y = |f(x)|$ for some cubic function $f(x)$.



a On separate diagrams sketch the two possible graphs of $y = f(x)$.

b Briefly explain how to distinguish between these two possibilities.

4 The function $f(x)$ is periodic with a period of 4, and is defined by 4

$$f(x) = x^2 + 1 \text{ for } -1 \le x \le 1 \quad \text{and} \quad f(x) = 2 \text{ for } 1 \le x \le 3$$

a Sketch the graph of $y = f(x)$ for $-5 \le x \le 5$.

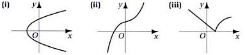
b Determine the values of $f(4.5)$ and $f(10)$.

Consolidation

Exercise A

1 **a** State whether or not each of graphs **i**, **ii** and **iii** represents a function.

(i) (ii) (iii)



b One of the graphs is such that the function f represented by the graph has an inverse f^{-1}. Assuming equal scales on the axes of the graphs drawn, sketch the graph of f^{-1}.

(AEB)

2 Functions f and g are defined for all real values of x by $f: x \to x^2$ and $g: x \to 4 - 9x$.

a Express the composite function gf in terms of x.
b Sketch the curve with equation $y = gf(x)$ and label the coordinates of the points at which your curve intersects the x-axis.
c Determine the range of the function gf.
d Find the value of x for which $g(x) = g^{-1}(x)$, where g^{-1} is the inverse function of g.

(ULEAC)

3 The functions f and g are defined by $f: x \to 9x^2 - 4$ for $\{x \in \mathbb{R}: x \geq 0\}$ and $g: x \to \sqrt{x+1}$, for $\{x \in \mathbb{R}: x \geq 0\}$.

a State the range of f.
b Sketch the graph of f and hence explain why the inverse function f^{-1} exists. Find f^{-1}, stating its domain.
c The composite function $f \circ g$ is defined for $x \geq 0$.
  i Find $f \circ g(x)$.
  ii Determine the exact surd solution to the equation $f(x) = f \circ g(x)$.

(AEB)

$f \circ g$ is alternative notation for the composite function $fg(x)$.

4 The function f has as its domain the set of all non-zero real numbers, and is given by $f(x) = \frac{1}{x}$ for all x in this set. On a single diagram, sketch the following graphs, and indicate the geometrical relationships between them:

a $y = f(x)$ **b** $y = f(x+1)$ **c** $y = f(x+1) + 2$

Deduce, explaining your reasoning, the coordinates of the point about which the graph of $y = \dfrac{2x+3}{x+1}$ is symmetrical.

(UCLES)

5 Function f is defined on the domain $0 \le x \le \frac{\pi}{8}$ by $f(x) = \tan 2x$. Find an expression for $f^{-1}(x)$ and state, or obtain, the domain of f^{-1}.

(NEAB)

6 Functions f and g are defined by $f\colon x \to 4 - x$ for $\{x \in \mathbb{R}\}$ and $g\colon x \to 3x^2$ for $\{x \in \mathbb{R}\}$.

a Find the range of g.
b Solve $gf(x) = 48$.
c Sketch the graph of $y = |f(x)|$ and hence find the values of x for which $|f(x)| = 2$.

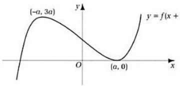
(ULEAC)

7 Functions f and g are defined by $f\colon x \to 3x - 1$ for $\{x \in \mathbb{R}\}$ and $g\colon x \to x^2 + 1$ for $\{x \in \mathbb{R}\}$.

a Find the range of g.
b Determine the values of x for which $gf(x) = fg(x)$.
c Determine the values of x for which $|f(x)| = 8$.
d Function $h\colon x \to x^2 + 3x$ for $\{x \in \mathbb{R}, x \ge q\}$ is one-to-one. Find the least value of q and sketch the graph of this function.

(ULEAC)

8 The diagram shows the curve $y = f(x + a)$, where a is a positive constant. The maximum and minimum points on the curve are $(-a, 3a)$ and $(a, 0)$ respectively.



Sketch the following curves, on separate diagrams, in each case stating the coordinates of the maximum and minimum points:

a $y = f(x)$ **b** $y = -2f(x + a)$.

(UCLES)

9 The function f is defined on the domain $x > 0$ by $f(x) = 1 + \frac{2}{x}$.

a **i** Find an expression for $f^{-1}(x)$.
ii State the domain of f^{-1}.

The composite function g is defined by $g = ff$.

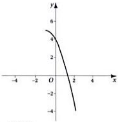
b **i** Find an expression for $g(x)$.
ii State the range of g.

(WJEC)

10 Function f is defined on the domain $-1 \le x \le 2$ by $f(x) = 4 - 2x - x^2$.

a Determine the values of a and b such that $f(x) = a - (b + x)^2$.

The diagram shows a sketch of the graph of $y = f(x)$.



b Describe, as either a single transformation or as two separate transformations, how the graph of $y = f(x)$ may be obtained from part of the graph of $y = -x^2$, $\{x \in \mathbb{R}\}$.

c State the coordinates of the points of $y = -x^2$ which correspond to the end points of $y = f(x)$.

d Sketch the graph of $y = f^{-1}(x)$.

e Calculate the value of x for which $f(x) = f^{-1}(x)$.

(NEAB)

Exercise B

1 The figure shows part of the graph of $y = \sqrt{x}$ (the scales on the two axes being the same). Describe the transformations that map this graph on to the graphs of:

a $y = 50 + \sqrt{x}$ **b** $y = \sqrt{x + 50}$ **c** $y = -\sqrt{x}$

d $x = \sqrt{y}$ **e** $y = 30\sqrt{x}$

(OCSEB)

2 Functions f and g are defined by $f(x) = 1 + x^2$ for $\{x \in \mathbb{R}\}$ and $g(x) = \tan x$ for $\{x \in \mathbb{R}: -\frac{\pi}{2} < x < \frac{\pi}{2}\}$.

a Write down $fg(x)$ and state the domain of fg.

b Find $g^{-1}(\sqrt{3})$ in terms of π.

c Explain briefly why f does not have an inverse.

(NEAB)

3 Functions f, g and h are defined for $\{x \in \mathbb{R}: x > 0\}$ by $f: x \to x^2$, $g: x \to \frac{2}{x}$ and $h: x \to \sqrt{x}$. Express in terms of x:

a $gh(x)$ **b** $fgh(x)$

State which two of the three functions f, g and h are inverses of each other.

(UCLES)

4 Functions f and g are defined with their respective domains by $f: x \to \frac{3}{(2x-1)}$ for $\{x \in \mathbb{R}, x \neq \frac{1}{2}\}$ and $g: x \to x^2 + 1$ for $\{x \in \mathbb{R}\}$.

a Find the values of x for which $f(x) = x$.
b Find the range of g.
c The domain of the composite function $f \circ g$ is $\mathbb{R}$. Find $f \circ g(x)$ and state the range of $f \circ g$.

(AEB)

Recall the alternative notation for the composite function $fg(x)$.

5 Function f is defined on the domain $0 \leq x \leq 2$ by $f(x) = x^2$. Function g is defined by translating the graph of $y = f(x)$, with this domain, 3 units in the positive x direction and 5 units in the positive y direction to give the graph of $y = g(x)$.

a Sketch the graphs of $y = f(x)$ and $y = g(x)$.
b State the domain and range of function g.
c Find an expression for $g(x)$.
d Find an expression for $g^{-1}(x)$.
e State the domain and range of the function g^{-1}.

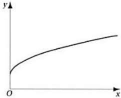
(NEAB)

6 Express $x^2 + 4x$ in the form $(x + a)^2 + b$, stating the numerical values of a and b. Functions f and g are defined as $f: x \to x^2 + 4x$ for $\{x \in \mathbb{R}: x > -2\}$ and $g: x \to x + 6$ for $\{x \in \mathbb{R}\}$.

a Show that the equation $gf(x) = 0$ has no real roots.
b State the domain of f^{-1}.
c Find an expression in terms of x for $f^{-1}(x)$.
d Sketch, on a single diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

(UCLES)

7 The diagram shows the graph of the function f defined for $x \geq 0$ by $f: x \to 1 + \sqrt{x}$.

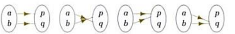


a Copy the sketch, and show on the same diagram the graph of f^{-1}, making clear the relationship between the two graphs.
b Give an expression in terms of x for $f^{-1}(x)$, and state the domain of f^{-1}.
c There is one value of x for which $f(x) = f^{-1}(x)$. By considering your diagram, explain why this value of x satisfies the equation $1 + \sqrt{x} = x$.
d By treating the equation $1 + \sqrt{x} = x$ as a quadratic equation for $\sqrt{x}$, or otherwise, show that the value of x satisfying $f(x) = f^{-1}(x)$ is $x = \frac{1}{2}(3 + \sqrt{5})$.

(UCLES)

Applications and Activities

1 There are four possible functions that map elements in the set $\{a, b\}$ across to elements in the set $\{p, q\}$, as illustrated by the mapping diagrams below.



In the first two mappings, $\{a, b\}$ is the domain and $\{p, q\}$ is the range, containing all the output values. However, in the second two mappings, $\{p, q\}$ is the co-domain of the functions, containing the possible output values.

Now consider the functions that could possibly map elements in $\{a, b, c\}$ to elements in $\{p, q\}$.

a How many have $\{p, q\}$ as their range, and how many have $\{p, q\}$ as their co-domain?

b What is the total number of possible functions?

c How many different functions exist in the more general case of m elements in the domain and n elements in the co-domain?

Summary

- A **function** is a many-to-one or one-to-one mapping for which each input value in the **domain** gives only one output value.
- The **range** of a function is the set of output values to which values in its domain map; the range is a subset of the co-domain.
- Only functions which, for a given domain, are one-to-one mappings, have inverse functions.
- The graph of the inverse function $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ in the line $y = x$ for values of x where f is one-to-one.
- An algebraic expression for $f^{-1}(x)$ can be found by interchanging x and y in the equation $y = f(x)$ and then rearranging it to make y the subject of the equation again.
- Functions for which $f^{-1}(x) \equiv f(x)$ are **self-inverses.**
- The inverse trigonometric functions $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} x$ only exist for the restricted domains in which the sine, cosine and tangent functions are one-to-one.

- The composite function $fg(x)$ (alternative notation $f \circ g(x)$) is obtained by putting values of $g(x)$ into function f.
- The composite function $ff(x)$, or $f^2(x)$, is obtained by putting values of $f(x)$ back into the function f again.
- The graph of $y = f(x)$ can, by a combination of translations, stretches, and reflections, be transformed into the graph of a related function, as shown in this table.

Function	Transformation
$f(x) + a$	translation of $+a$ units parallel to the y-axis
$f(x + a)$	translation of $-a$ units parallel to the x-axis
$af(x)$	one-way stretch, of factor a, parallel to the y-axis
$f(ax)$	one-way stretch, of factor $\frac{1}{a}$, parallel to the x-axis
$-f(x)$	reflection in the x-axis
$f(-x)$	reflection in the y-axis

- **Even** functions are those that satisfy the condition $f(-x) = f(x)$, and are therefore symmetrical about the y-axis.
- **Odd** functions are those that satisfy the condition $f(-x) = -f(x)$, and therefore have rotational symmetry about the origin.
- **Periodic** functions are those for which $f(x \pm k) \equiv f(x)$ for some constant k; such functions repeat themselves every k units along the x-axis, and k is the period of the function.

5 Differentiation I

What you need to know

- How to find the gradient of the straight line joining two points.
- How to find the value of a function.
- How to find the equation of the straight line that passes through a given point, when its gradient is known.
- How to solve simple trigonometric equations.
- How to use negative and fractional indices.

Review

1 Find the gradient of the straight line joining the following pairs of points:

a $(2, 1)$ and $(6, 9)$ **b** $(0, 7)$ and $(3, 1)$
c $(-2, 11)$ and $(1, -7)$ **d** $(-1, -2)$ and $(-4, -11)$

2 Find the value of the following functions at the given value of x:

a $f(x) = x^2 + 5x + 6$, when $x = -4$
b $f(x) = 3x + \frac{1}{x^2}$, when $x = \frac{1}{2}$
c $f(x) = \sin 2x - 2\cos^2 x$, when $x = \frac{\pi}{4}$

3 Find the equation of the straight line that passes through the given point with the gradient indicated, giving your answer in the form $y = mx + c$:

a $(1, 4)$, gradient 6 **b** $(3, -11)$, gradient -3
c $(-5, 2)$, gradient $-\frac{1}{4}$ **d** $(2, 3)$, gradient $\frac{2}{3}$

4 Solve the following trigonometric equations for $0 \leq \theta \leq 2\pi$:

a $\tan 2\theta = \sqrt{3}$ **b** $\sin\theta + \cos\theta = 0$
c $\sin 2\theta = \cos\theta$ **d** $\cos 2\theta + \cos\theta = 0$

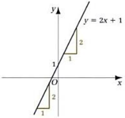
5 Write each of the following in index notation:

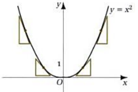
a $\dfrac{2}{x^3}$ **d** $\dfrac{10}{\sqrt{x}}$
b $\sqrt{x^3}$ **e** $(\sqrt[3]{x})^4$
c $\dfrac{1}{x+5}$

5.1 Finding the Gradient of a Curve

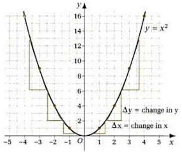
The gradient at each point on the graph of a linear function, such as $y = 2x + 1$, is constant. However, many mathematical functions are not linear, and have curved graphs whose gradients are continuously changing.

Look at the graph of $y = x^2$. Notice that the gradient changes from being negative to positive as the graph crosses the y-axis. It also becomes steeper for larger positive and negative values of x.





Finding the gradient of the curve is not as straightforward as it is for the linear function. The gradient of the curved graph at any particular point can be found by calculating the gradient of the tangent to the curve at this point. Look at the graph of $y = x^2$ again in more detail.



Draw tangents to this graph at the points $x = -3, -2, -1, 0, 1, 2, 3$. Notice that for $x > 0$, the gradient is positive and increases as the tangents become steeper. When $x < 0$, the gradient is negative, and becomes more negative as x decreases. By constructing right-angled triangles and using

$$\text{gradient} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

The Greek letter Δ is used to mean 'a change in'.

we can estimate the gradient of the curve at these different points.

x	-3	-2	-1	0	1	2	3
Gradient of the curve $y = x^2$	-6	-4	-2	0	2	4	6

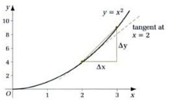
These results suggest that the **gradient function** of the curve $y = x^2$ is $2x$. This means that the gradient at any point on this curve can be calculated by multiplying the x-coordinate by 2. The gradient function describes algebraically how the gradient is changing.

However this method of finding the gradient function, by drawing tangents to the curve at a number of different points and then calculating their gradients, has a number of drawbacks.

- Its accuracy depends on the accuracy with which the graph and the tangents to it are drawn, and on the accuracy of the measurements of Δx and Δy.
- It cannot easily be translated into an algebraic procedure.

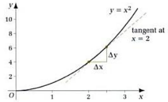
How else could the gradient function be found? An alternative method would be to draw a chord from a particular point on a curve to some nearby point. For example, the gradient of the curve $y = x^2$ at $x = 2$ can be estimated by finding the gradient of the chord drawn from $x = 2$ to $x = 3$ on the curve. The gradient of this chord can be calculated using,

$$\begin{aligned}\text{gradient of chord} &= \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}\\ &= \frac{3^2 - 2^2}{3 - 2}\\ &= \frac{9 - 4}{1}\\ &= 5\end{aligned}$$



The gradient of this chord is only a rough approximation to the gradient of the tangent, and to the gradient of the curve itself at $x = 2$. A more accurate value can be found using the chord between the points on the graph corresponding to $x = 2$ and $x = 2.5$.

$$\begin{aligned}\text{gradient of chord} &= \frac{\Delta y}{\Delta x}\\ &= \frac{2.5^2 - 2^2}{2.5 - 2}\\ &= \frac{6.25 - 4}{0.5}\\ &= 4.5\end{aligned}$$



Notice that the smaller the change in x (denoted Δx) over which this chord is drawn, the closer its gradient will approximate that of the tangent to the curve at $x = 2$.

Try using $\Delta x = 0.1$.

$$\begin{aligned}\text{gradient of chord} &= \frac{\Delta y}{\Delta x}\\ &= \frac{2.1^2 - 2^2}{2.1 - 2}\\ &= \frac{4.41 - 4}{0.1}\\ &= 4.1\end{aligned}$$

Repeat this procedure with $\Delta x = 0.01$, and tabulate all your results. What do you notice?

Δx	1	0.5	0.1	0.01
Gradient of chord drawn from $x = 2$	5	4.5	4.1	4.01

From these results, notice that as Δx **tends towards** zero (written '$\Delta x \to 0$'), the gradient of the chord tends towards a value of 4. Try $\Delta x = 0.001$ at $x = 2$. Does your result get closer to the value of 4?

Find the gradient of the curve $y = x^2$ at the points $x = -3, -2, -1, 0, 1,$ and 3 by calculating the gradient of the chords drawn from these points, using $\Delta x = 1, 0.1,$ and 0.01. Notice that as $\Delta x \to 0$, the gradients of these chords converge towards a **limiting value**. These results also suggest that the gradient function of the curve $y = x^2$ is $2x$.

Graphi
calcula
support
pack

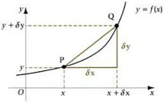
Check to see if your graphical calculator l a derivative function key. This will give th numerical value of th gradient of a curve at chosen point.

This method of finding the gradient of a curve by drawing chords between two nearby points on it has two major advantages.

- It does not rely on the accurate drawing of a tangent to the curve, and measurement of Δy and Δx because Δy can be calculated for any given value of Δx.
- It can be translated into an algebraic procedure, known as **differentiation from first principles**.

Differentiation from first principles

Consider the point P with coordinates (x, y) on the curve $y = f(x)$. Let the nearby point Q on the curve have coordinates $(x + \delta x, y + \delta y)$, where δx is the *small* change in the value of the x-coordinate between P and Q, and δy is the corresponding *small* change in the value of the y-coordinate.

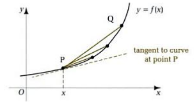


The gradient of the chord PQ is given by

$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}} = \frac{\delta y}{\delta x} = \frac{(y + \delta y) - y}{(x + \delta x) - x}$$

$$= \frac{f(x + \delta x) - f(x)}{\delta x}$$

If $y = f(x)$ then $y + \delta y = f(x + \delta x)$



For smaller values of δx, point Q approaches point P. The gradient of the chord PQ becomes closer to the gradient of the tangent to the curve at point P. So, in the limiting case, as $\delta x \to 0$,

$$\text{gradient of the curve at point P} = \underset{\delta x \to 0}{\text{limit}} \left(\frac{\delta y}{\delta x}\right)$$
$$= \underset{\delta x \to 0}{\text{limit}} \left[\frac{f(x+\delta x) - f(x)}{\delta x}\right]$$

This process of finding the gradient of a curve at some point P (x, y) by calculating the gradient of the chord to the point Q $(x + \delta x, y + \delta y)$, as δx tends towards zero, is called differentiation from first principles.

The resulting **gradient function** of the curve is denoted $\frac{dy}{dx}$ or $f'(x)$, and is a function of x.

$$\frac{dy}{dx} = f'(x) = \underset{\delta x \to 0}{\text{limit}} \left[\frac{f(x+\delta x) - f(x)}{\delta x}\right]$$

Note that $\frac{dy}{dx}$ is *not* a fraction. It refers to the gradient of the graph of $y = f(x)$.

Example 1

Using differentiation from first principles, show that the gradient function of the curve $y = x^2$ is $\frac{dy}{dx} = 2x$.

Solution

Let $y = f(x)$, where $f(x) = x^2$. A small change δx in the x-coordinate of some point on the curve $y = x^2$ will result in a corresponding small change δy in its y-coordinate, such that

$$\begin{aligned}\delta y &= f(x+\delta x) - f(x)\\ &= (x+\delta x)^2 - x^2\\ &= x^2 + 2x\,\delta x + (\delta x)^2 - x^2\\ &= 2x\,\delta x + (\delta x)^2\end{aligned}$$

The gradient function of the curve is given by

$$\begin{aligned}\frac{dy}{dx} &= \underset{\delta x \to 0}{\text{limit}} \left(\frac{\delta y}{\delta x}\right)\\ &= \underset{\delta x \to 0}{\text{limit}} \left[\frac{2x\,\delta x + (\delta x)^2}{\delta x}\right]\\ &= \underset{\delta x \to 0}{\text{limit}}\,(2x + \delta x) \quad \blacktriangleleft \textbf{ As } \delta x \to 0, (2x + \delta x) \to 2x\\ &= 2x\end{aligned}$$

So, the gradient at any point on the curve $y = x^2$ can be calculated using the gradient function $\frac{dy}{dx} = 2x$.

Example 2

Find the gradient function for the general quadratic curve $y = ax^2 + bx + c$, where a, b and c are constants.

Solution

Let $y = f(x)$, where $f(x) = ax^2 + bx + c$. The small change δy in the value of y that results from a small change δx in the value of x is given by

$$\begin{aligned}\delta y &= f(x + \delta x) - f(x) \\ &= [a(x + \delta x)^2 + b(x + \delta x) + c] - [ax^2 + bx + c] \\ &= ax^2 + 2ax\,\delta x + a(\delta x)^2 + bx + b\,\delta x + c - ax^2 - bx - c \\ &= 2ax\,\delta x + a(\delta x)^2 + b\,\delta x\end{aligned}$$

The gradient function of the curve is given by

$$\begin{aligned}\frac{dy}{dx} &= \underset{\delta x \to 0}{\text{limit}} \left(\frac{\delta y}{\delta x}\right) \\ &= \underset{\delta x \to 0}{\text{limit}} \left[\frac{2ax\,\delta x + a(\delta x)^2 + b\,\delta x}{\delta x}\right] \\ &= \underset{\delta x \to 0}{\text{limit}} \,(2ax + a\delta x + b) \\ &= 2ax + b\end{aligned}$$

So, the gradient function at any point on a quadratic curve of the form $y = ax^2 + bx + c$ is given by $\frac{dy}{dx} = 2ax + b$.

Differentiating $y = x^n$

Differentiating from first principles is a way of establishing the gradient function (or **derivative**) of different types of functions, but this process can be tedious. The derivative of functions of the form $y = x^n$, where n is a positive integer, all follow a similar pattern.

Remember that the process of finding the gradient function is known as **differentiation**.

Function	$y = x^1$	$y = x^2$	$y = x^3$	$y = x^4$	$y = x^5$
Derivative	1	$2x$	$3x^2$	$4x^3$	$5x^4$

$y = x$ is a straight line of gradient 1.

Can you spot the similarities in the derivatives? Notice that the following algebraic rule explains the similarities.

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$ ◀ **Learn this result.**

Expressed in words, this rule becomes 'multiply by the power, and *then* reduce the power by one'. A more general rule for differentiating powers of x, is:

These rules also hold for fractional and negative powers.

If $y = ax^n$, where a is a constant, then $\frac{dy}{dx} = nax^{n-1}$

◀ **Learn this result.**

Example 3

Differentiate the following with respect to x:

a $y = x^{10}$ **b** $y = 5x^6$ **c** $y = \sqrt{x}$

d $y = \dfrac{3}{x^2}$ **e** $y = \dfrac{1}{\sqrt[3]{x}}$ **f** $y = 4$

Solution

a $y = x^{10}$

$$\frac{dy}{dx} = 10x^{10-1} = 10x^9$$

Remember, multiply by the power, 10, and then reduce the power by 1, $10 - 1 = 9$.

b $y = 5x^6$

$$\frac{dy}{dx} = 6 \times 5x^{6-1} = 30x^5$$

c $y = \sqrt{x} = x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Express the function in the form $y = ax^n$ before differentiating; that is, convert roots to powers.

Recall, $x^{-n} = \dfrac{1}{x^n}$.

d $y = \dfrac{3}{x^2} = 3x^{-2}$

$$\frac{dy}{dx} = -2 \times 3x^{-2-1}$$

$$= -6x^{-3} = -\frac{6}{x^3}$$

e $y = \dfrac{1}{\sqrt[3]{x}} = \dfrac{1}{x^{\frac{1}{3}}} = x^{-\frac{1}{3}}$

$$\frac{dy}{dx} = -\frac{1}{3}x^{-\frac{1}{3}-1}$$

$$= -\frac{1}{3}x^{-\frac{4}{3}} = -\frac{1}{3x^{\frac{4}{3}}}$$

$-\dfrac{1}{3x^{\frac{4}{3}}}$ can also be written $-\dfrac{1}{3\sqrt[3]{x^4}}$ or $\dfrac{-1}{3(\sqrt[3]{x})^4}$.

f $y = 4$ can be expressed $y = 4x^0$, since $x^0 = 1$

$$\frac{dy}{dx} = 0 \times 4x^{0-1} = 0$$

This is as expected, because the graph of $y = 4$ is horizontal; its gradient is 0.

By differentiating from first principles, it can be shown that the derivative of the sum (or difference) of two or more functions is simply the sum (or difference) of their individual derivatives. That is,

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{df}{dx} \pm \frac{dg}{dx} = f'(x) \pm g'(x)$$

◀ **Learn this result.**

Example 4

Find the derivatives of the following functions of x:

a $y = x^3 + 4x^2 - 9x - 13$ **b** $y = 5x + \frac{1}{x} - \frac{2}{x^3}$

c $y = (2x - 1)(x + 3)$ **d** $y = \frac{x^3 + 1}{x^2}$

Solution

a $y = x^3 + 4x^2 - 9x - 13$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(4x^2) - \frac{d}{dx}(9x) - \frac{d}{dx}(13)$$

$$= 3x^2 + 8x - 9 - 0$$

$$= 3x^2 + 8x - 9$$

Differentiating each term separately.

b $y = 5x + \frac{1}{x} - \frac{2}{x^3}$

$$= 5x + x^{-1} - 2x^{-3}$$

Remember to express each term as a power of x before differentiating.

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(x^{-1}) - \frac{d}{dx}(2x^{-3})$$

$$= 5 + (-1 \times x^{-1-1}) - (-3 \times 2x^{-3-1})$$

$$= 5 - x^{-2} + 6x^{-4}$$

$$= 5 - \frac{1}{x^2} + \frac{6}{x^4}$$

Take care with the signs when multiplying by negative numbers.

c $y = (2x - 1)(x + 3)$

$$= 2x^2 + 5x - 3$$

$$\frac{dy}{dx} = 4x + 5$$

Expand brackets and collect like terms before differentiating.

d $y = \frac{x^3 + 1}{x^2}$

$$= \frac{x^3}{x^2} + \frac{1}{x^2} = x + x^{-2}$$

Separate the fraction into two terms.

$$\frac{dy}{dx} = 1 + (-2x^{-3}) = 1 - 2x^{-3}$$

$$= 1 - \frac{2}{x^3}$$

$$= \frac{x^3 - 2}{x^3}$$

The last step simply puts the answer in the form of the question, as a single fraction, using the common denominator x^3.

Differentiation has so far been mainly confined to functions of the form $y = f(x)$, where $f(x) = ax^n$. The rules for differentiating can be applied when other letters are used. For example, if $z = g(t)$, where $g(t) = at^n$ and a is a constant, we can differentiate z 'with respect to' t. The derivative $\frac{dz}{dt} = g'(t) = nat^{n-1}$.

Example 5

Find the derivatives of the following functions:

a $s = ut + \frac{1}{2}at^2$, where u and a are constant

b $p = \theta - \dfrac{\theta^3}{6} + \dfrac{\theta^5}{120}$

c $\theta = \dfrac{2A}{r^2}$, where A is constant.

Solution

a $s = ut + \frac{1}{2}at^2$

The variable on the RHS is t. Both u and a are constant.

$$\frac{ds}{dt} = u + (2 \times \tfrac{1}{2}at^{2-1}) = u + at$$

Differentiating s with respect to t.

b $p = \theta - \dfrac{\theta^3}{6} + \dfrac{\theta^5}{120}$

The variable on the RHS is θ.

$$\frac{dp}{d\theta} = 1 - \frac{3\theta^2}{6} + \frac{5\theta^4}{120}$$

$$= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$$

Differentiating p with respect to θ.

c $\theta = \dfrac{2A}{r^2} = 2Ar^{-2}$

The variable is r.

$$\frac{d\theta}{dr} = -4Ar^{-3} = -\frac{4A}{r^3}$$

Differentiating θ with respect to r.

Rates of change

The derivative, $\frac{dy}{dx}$ or $f'(x)$, describes the rate at which the value of y changes with respect to x at different points on the curve. The rate of change at any particular value of x can be found by substituting this value into the expression derived for $\frac{dy}{dx}$. For example, $f'(2)$ is the rate at which the function $y = f(x)$ is changing with respect to x when $x = 2$.

The concept of 'rate of change' can be applied to equations in other variables. Consider the volume of a sphere, given by $V = \frac{4}{3}\pi r^3$. The derivative with respect to variable r, $\frac{dV}{dr} = 4\pi r^2$, gives the rate at which the volume V changes with respect to the radius r at any given value of r.

Example 6

Find the rate of change with respect to the given variable of the following functions at the values indicated:

a $f(x) = x^2 - 7x$, when $x = 3$

b $u(\theta) = (\theta^2 - 1)(\theta + 1)$, when $\theta = \frac{1}{3}$.

Solution

a $f(x) = x^2 - 7x$

$\Rightarrow f'(x) = 2x - 7$

Substituting $x = 3$, $f'(3) = 6 - 7$

$= -1$

b $u(\theta) = (\theta^2 - 1)(\theta + 1)$

$= \theta^3 + \theta^2 - \theta - 1$

$\Rightarrow u'(\theta) = 3\theta^2 + 2\theta - 1$

Substituting $\theta = \frac{1}{3}$, $u'(\frac{1}{3}) = 3 \times \frac{1}{9} + 2 \times \frac{1}{3} - 1$

$= \frac{3}{9} + \frac{2}{3} - 1 = 0$

Expand brackets firs

Differentiate u with respect to θ.

Example 7

Find the coordinates of the point(s) on the following curves at which the gradient has the value indicated:

a $y = x^2 + 4x + 1$, $\dfrac{dy}{dx} = 8$

b $y = \dfrac{1}{x} - 2x$, $\dfrac{dy}{dx} = -6$

Solution

a $y = x^2 + 4x + 1 \Rightarrow \dfrac{dy}{dx} = 2x + 4$

If $\dfrac{dy}{dx} = 8$, then $2x + 4 = 8$

This gives $2x = 4 \Rightarrow x = 2$

$\Rightarrow y = (2)^2 + (4 \times 2) + 1 = 13$

So, the gradient of the curve $y = x^2 + 4x + 1$ is 8 at the point (2, 13).

Substituting the val x in the original equation.

b $y = \dfrac{1}{x} - 2x = x^{-1} - 2x$

$\Rightarrow \dfrac{dy}{dx} = -x^{-2} - 2 = -\dfrac{1}{x^2} - 2$

If $\dfrac{dy}{dx} = -6$, then $-\dfrac{1}{x^2} - 2 = -6$

$\Rightarrow \quad \dfrac{1}{x^2} = 4$

$\Rightarrow \quad x^2 = \frac{1}{4}$

$\Rightarrow \quad x = \frac{1}{2}$ or $x = -\frac{1}{2}$

When $x = \frac{1}{2}$, $y = \frac{1}{\left(\frac{1}{2}\right)} - 2 \times \left(\frac{1}{2}\right) = 2 - 1 = 1$

When $x = -\frac{1}{2}$, $y = \frac{1}{\left(-\frac{1}{2}\right)} - 2 \times \left(-\frac{1}{2}\right) = -2 + 1 = -1$

So the curve $y = \frac{1}{x} - 2x$ has a gradient of -6 at the points $\left(\frac{1}{2}, 1\right)$ and $\left(-\frac{1}{2}, -1\right)$.

Differentiating with respect to *y*

It can be shown that the rate of change of x with respect to y, $\frac{dx}{dy}$, is the reciprocal of $\frac{dy}{dx}$.

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

◀ Learn this result.

For example, if $y = 3x^2 + 2x$, then $\frac{dy}{dx} = 6x + 2$, and $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{6x + 2}$.

This particular method is quicker than having to rearrange $y = 3x^2 + 2x$ to express x in terms of y before differentiating with respect to y.

Higher derivatives

Sometimes it is useful to know the gradient of the gradient function at a particular point on the curve $y = f(x)$. This is called the **second derivative**. It is found by differentiating the first derivative, $\frac{dy}{dx}$, with respect to x to give $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, written $\frac{d^2y}{dx^2}$, or $f''(x)$.

The expression $\frac{d^2y}{dx^2}$ is read 'd-two-y-by-d-x-squared', and shows that the differentiation process has happened twice. It is *not* the same as squaring $\frac{dy}{dx}$.

It is important to remember that $\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2$.

Differentiating again with respect to x would give the third derivative, $\frac{d^3y}{dx^3}$, or $f'''(x)$.

$\frac{d^3y}{dx^3}$ is read 'd-three-y-by-d-x-cubed'.

Higher derivatives, of the form $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$, can be obtained by differentiating $y = f(x)$ n times with respect to x.

Example 8

Find the first, second and third derivatives of the following functions:

a $y = x^4 + 5x^3 - 9x^2 + 2x - 7$

b $f(x) = x^3 - \frac{1}{x}$

Solution

a

$$y = x^4 + 5x^3 - 9x^2 + 2x - 7$$

$$\Rightarrow \frac{dy}{dx} = 4x^3 + 15x^2 - 18x + 2$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right)\\ &= \frac{d}{dx}(4x^3 + 15x^2 - 18x + 2)\\ &= 12x^2 + 30x - 18\end{aligned}$$

$$\begin{aligned}\frac{d^3y}{dx^3} &= \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)\\ &= \frac{d}{dx}(12x^2 + 30x - 18)\\ &= 24x + 30\end{aligned}$$

b

$$\begin{aligned}f(x) &= x^3 - \frac{1}{x}\\ &= x^3 - x^{-1}\\ f'(x) &= 3x^2 + x^{-2}\\ &= 3x^2 + \frac{1}{x^2}\\ f''(x) &= 6x - 2x^{-3}\\ &= 6x - \frac{2}{x^3}\\ f'''(x) &= 6 + 6x^{-4}\\ &= 6 + \frac{6}{x^4}\end{aligned}$$

Remember to express each term in the form ax^n before differentiating.

5.1 Finding the Gradient of a Curve

Exercise

Technique

1 Differentiate each of the following from first principles to find $\frac{dy}{dx}$:

a $y = x^3$ **b** $y = 3x^2 - 9x + 5$ **c** $y = \frac{1}{x}$

2 Differentiate each of the following with respect to x:

a $y = x^8$

b $y = 7 + x - 3x^2 - \frac{1}{3}x^3$

c $y = (3x - 5)(2x + 1)$

d $y = \sqrt[4]{x}$

e $y = -\dfrac{4}{x^3}$

f $y = \dfrac{2 - x^3}{x^2}$

g $y = \dfrac{6}{\sqrt{x}}$

h $y = \dfrac{3x^2 - 2x}{\sqrt{x}}$.

2 b

3 Find the first derivative of each of the following functions:

a $f(x) = \frac{1}{4}x^2 - \frac{1}{2}x$

b $f(t) = (t + 1)(2 - t)$

c $f(s) = s^3 - 7s^2 - 2s$

d $h(p) = \dfrac{2p^4 - 5p}{p^3}$

e $\theta(t) = 6t^{\frac{1}{3}}$

4 Find the gradient of each of the following curves at the point indicated:

a $y = x^2 + 6x - 3$ at $(2, 13)$

b $y = 2x^3 - 7x - 5$ at $(-1, 0)$

c $y = \frac{4}{x} + \frac{x^2}{4}$ at $(2, 3)$

d $y = (x^2 - 2)(x + 1)$ at $(-3, -14)$

4 c

5 Find the coordinates of the points on each of the following curves at which the gradient has the value indicated:

a $y = x^3 - 6x^2 + 7x$, where $\frac{dy}{dx} = -2$

b $y = 3 - 5x + x^3$, where $\frac{dy}{dx} = 7$

c $y = 2x + 1 - \frac{4}{x^2}$, where $\frac{dy}{dx} = 1$

d $y = \frac{1}{x}$, where $\frac{dy}{dx} = -\frac{4}{9}$

5 c

6 Find the gradient of each of the following curves at their points of intersection with the x- and y-axes:

a $y = x^2 + 2x - 3$ **b** $y = (2x + 3)(x - 1)$

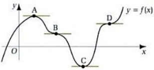
6 a

7 Find the first, second and third derivatives of each of the following:

a $y = 5x^4 + 2x^3 - 7x^2 - 9x + 2$ **b** $f(x) = \frac{4}{x} - \frac{1}{x^2}$

5.2 Stationary Points

Look at the graph below. What do points A, B, C and D have in common?



Notice that the gradient of the curve is zero at these four points. This means that $\frac{dy}{dx} = 0$ at A, B, C and D.

Points like A, B, C and D are known as **stationary points**. They correspond to values of x for which the gradient of the curve is zero. This means the function $f(x)$ is momentarily 'stationary', that is neither increasing or decreasing.

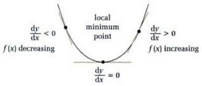
Points A and C are known as **turning points** on the graph. Notice that at a turning point $f(x)$ changes from being an increasing function of x to a decreasing function of x, or vice versa. Point A is called a **local maximum point** and point C is a **local minimum point**.

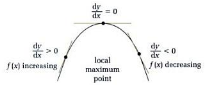
What about points B and D? $f(x)$ is a decreasing function on either side of point B. It is an increasing function on either side of point D, but the gradient is zero at both points. Such points are called **stationary points of inflexion**.

Colin Maclaurin
(1698–1746)
Maclaurin was the fi
present the corre
theory for distinguis
between the minim
and maximum value
a function.

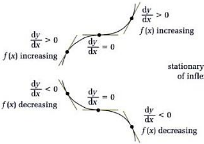
An alternative spelli
of 'inflexion' is
'inflection'.

Distinguishing between stationary points





As a graph passes through a local minimum point, its gradient function, $\frac{dy}{dx}$ changes from being negative to positive. It is zero at the minimum point itself. On either side of a local maximum point, there is a corresponding change in the sign of $\frac{dy}{dx}$ from positive to negative.



Since the curve is either an increasing or decreasing function of x on *both* sides of a stationary point of inflexion, its gradient $\frac{dy}{dx}$ remains either positive or negative on both sides of the point of inflexion. The gradient is zero at the point of inflexion itself.

So, the stationary points on a curve can be located by solving $\frac{dy}{dx} = 0$. Then their nature can be determined by examining the gradient of the curve on either side of the stationary point. The table below summarises the changes in the sign of $\frac{dy}{dx}$ and the corresponding type of stationary point.

You must find these points and their nature before you can make an accurate sketch of a curve.

Change in $\frac{dy}{dx}$	Type of stationary point
$-\text{ve} \rightarrow 0 \rightarrow +\text{ve}$	minimum
$+\text{ve} \rightarrow 0 \rightarrow -\text{ve}$	maximum
$+\text{ve} \rightarrow 0 \rightarrow +\text{ve}$	point of inflexion
$-\text{ve} \rightarrow 0 \rightarrow -\text{ve}$	point of inflexion

◀ **Learn these important results.**

Example 1

Find the coordinates of the points where the quadratic curve $y = x^2 - 2x - 15$ crosses the x- and y-axes. Find the coordinates of the stationary point on the curve, and determine its nature. Then, sketch the graph of $y = x^2 - 2x - 15$.

Solution

The graph crosses the x-axis when $y = 0$.

$$\Rightarrow \quad x^2 - 2x - 15 = 0$$
$$\Rightarrow \quad (x+3)(x-5) = 0$$
$$\Rightarrow \quad x = -3 \text{ or } x = 5$$

So the graph crosses the x-axis at $(-3, 0)$ and $(5, 0)$.

It crosses the y-axis when $x = 0 \Rightarrow y = -15$. So the graph crosses the y-axis at $(0, -15)$.

The gradient function for the graph of $y = x^2 - 2x - 15$ is given by $\frac{dy}{dx} = 2x - 2$. To find the location of the stationary point on this graph, solve $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = 0 \Rightarrow 2x - 2 = 0$$
$$\Rightarrow \quad x = 1$$

Now substitute this value into the equati[on] of the graph to find t[he] y-coordinate.

When $x = 1$, $y = (1)^2 - 2(1) - 15 = -16$.

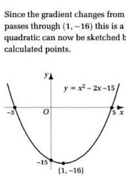
So, the stationary point is at $(1, -16)$.

Now look at $\frac{dy}{dx}$ on either side of $x = 1$.

When $x = 0$, $\dfrac{dy}{dx} = -2$. That is, $\dfrac{dy}{dx} < 0$.

When $x = 2$, $\dfrac{dy}{dx} = 2$. That is, $\dfrac{dy}{dx} > 0$.

Since the gradient changes from being negative to positive as the graph passes through $(1, -16)$ this is a local minimum point. The graph of the quadratic can now be sketched by drawing a parabola through the four calculated points.



Example 2

For each of the following cubic functions, find the coordinates of any stationary points, and determine their nature. Use a graphical calculator to plot the graphs of these functions. Use the TRACE facility to *check* your results.

a $y = x^3 - 3x^2 - 9x + 10$ **b** $y = x^3 - 3x^2 + 3x + 1$

Solution

a The gradient function for the graph of $y = x^3 - 3x^2 - 9x + 10$ is given by $\frac{dy}{dx} = 3x^2 - 6x - 9$. To find the location of any stationary points on this graph, solve $\frac{dy}{dx} = 0$.

$$3x^2 - 6x - 9 = 0$$

$$\Rightarrow \quad 3(x^2 - 2x - 3) = 0$$

$$\Rightarrow \quad 3(x + 1)(x - 3) = 0$$

$$\Rightarrow \quad x = -1 \text{ or } x = 3$$

When $x = -1$, $y = (-1)^3 - 3(-1)^2 - 9(-1) + 10$

$= -1 - 3 + 9 + 10 = 15$

When $x = 3$, $y = (3)^3 - 3(3)^2 - 9(3) + 10$

$= 27 - 27 - 27 + 10 = -17$

So, the graph of $y = x^3 - 3x^2 - 9x + 10$ has stationary points at $(-1, 15)$ and $(3, -17)$.

Now look at the sign of $\frac{dy}{dx}$ on both sides of $x = -1$.

When $x = -2$, $\dfrac{dy}{dx} = 3(-2)^2 - 6(-2) - 9 = 15$. That is, $\dfrac{dy}{dx} > 0$.

To the left of the stationary point the function is increasing.

When $x = 0$, $\dfrac{dy}{dx} = 3(0)^2 - 6(0) - 9 = -9$. That is, $\dfrac{dy}{dx} < 0$

To the right of the stationary point the function is decreasing.

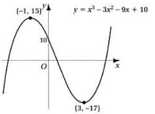
Since the gradient changes from positive to negative as the graph passes through $(-1, 15)$, this is a local maximum point.

Now look at the sign of $\frac{dy}{dx}$ on both sides of $x = 3$.

When $x = 2$, $\dfrac{dy}{dx} = 3(2)^2 - 6(2) - 9 = -9$. That is, $\dfrac{dy}{dx} < 0$.

When $x = 4$, $\dfrac{dy}{dx} = 3(4)^2 - 6(4) - 9 = 15$. That is, $\dfrac{dy}{dx} > 0$.

Since the gradient changes from negative to positive, $(3, -17)$ is a local minimum point on the curve.



The y-intercept is fou by substituting $x = 0$ into the equation of t curve.

b Any stationary points on the graph of $y = x^3 - 3x^2 + 3x + 1$ are found by solving $\frac{dy}{dx} = 0$.

Graphi calcula suppor pack

Since $\quad \dfrac{dy}{dx} = 3x^2 - 6x + 3$

then $\quad \dfrac{dy}{dx} = 0 \Rightarrow 3x^2 - 6x + 3 = 0$

$$\Rightarrow 3(x^2 - 2x + 1) = 0$$

$$\Rightarrow \quad 3(x-1)^2 = 0$$

$$\Rightarrow \quad x = 1$$

Note the repeated ro

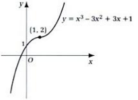
When $x = 1, \quad y = (1)^3 - 3(1)^2 + 3(1) + 1 = 2$.

So the only stationary point on this curve is located at $(1, 2)$.
To determine its nature, look at the sign of $\frac{dy}{dx}$ on either side of $x = 1$.

When $x = 0$, $\dfrac{dy}{dx} = 3$. That is, $\dfrac{dy}{dx} > 0$.

When $x = 2$, $\dfrac{dy}{dx} = 3$. That is, $\dfrac{dy}{dx} > 0$.

So the gradient is positive on both sides of $(1, 2)$, which must be a stationary point of inflexion.



Increasing and decreasing functions

We are sometimes more interested in whether a function is increasing or decreasing at various points along its curve. At points on the graph of $y = f(x)$ where the gradient is not zero, y must be either an increasing or decreasing function of x.

- If $\frac{dy}{dx} > 0$ (a positive gradient), then y is an increasing function of x.
- If $\frac{dy}{dx} < 0$ (a negative gradient), then y is a decreasing function of x.

For continuous functions, sections of increasing and decreasing behaviour are always separated by a stationary point. In order to find out for which values of x a function is increasing or decreasing, its stationary point(s) must first be located.

Example 3

Find the values of x for which the function $y = x^3 - 9x^2 + 15x + 13$ is increasing.

Solution

$$y = x^3 - 9x^2 + 15x + 13$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 18x + 15$$

At stationary values of function y, $\frac{dy}{dx} = 0$.

$$\Rightarrow 3x^2 - 18x + 15 = 0$$

$$\Rightarrow 3(x^2 - 6x + 5) = 0$$

$$\Rightarrow 3(x - 1)(x - 5) = 0$$

$$\Rightarrow \quad x = 1 \text{ or } x = 5$$

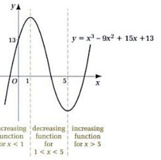
To determine the nature of these two stationary values, look at the gradient of $\frac{dy}{dx}$ on both sides.

When $x = 0$, $\frac{dy}{dx} = 15$. That is, $\frac{dy}{dx} > 0$.

When $x = 3$, $\frac{dy}{dx} = -12$. That is, $\frac{dy}{dx} < 0$.

When $x = 6$, $\frac{dy}{dx} = 15$. That is, $\frac{dy}{dx} > 0$.

This means that y has a maximum value at $x = 1$ and a minimum value at $x = 5$. It follows that y will be a decreasing function in the interval $1 < x < 5$, and an increasing function of x for $x < 1$ and $x > 5$.

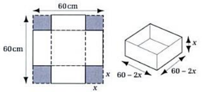


Optimisation

Many practical problems involve finding the maximum or minimum values of a function as it changes with respect to a particular variable. If the problem can be modelled in terms of a *mathematical* function, differentiation can be used to locate and distinguish between any maximum and minimum values.

Example 4

An open-topped gift box is formed by cutting squares of side length x from each corner of a 60 cm × 60 cm square of cardboard, and folding the remaining flaps to make the vertical sides.



a Find an expression in terms of x for the volume V (in cm^3) of the gift box.

b Use differentiation to find the value of x that corresponds to the maximum possible volume of the gift box.

c Calculate this maximum volume.

Solution

a The volume of the folded box is given by

V = length of box × width of box × height of box

and the dimensions of the box are

length = width = $60 - 2x$ and height = x

So $V = (60 - 2x) \times (60 - 2x) \times x$

$= x(60 - 2x)^2$

$V = 4x^3 - 240x^2 + 3600x$

b Differentiating with respect to x, the gradient of this volume function is given by

$$\frac{dV}{dx} = 12x^2 - 480x + 3600$$

At the maximum and minimum volumes, $\frac{dV}{dx} = 0$

$\Rightarrow \quad 12x^2 - 480x + 3600 = 0$

$\Rightarrow \quad 12(x^2 - 40x + 300) = 0$

$\Rightarrow \quad 12(x - 10)(x - 30) = 0$

$\Rightarrow \quad x = 10$ or $x = 30$

Look at the sign of $\frac{dV}{dx}$ on both sides of $x = 10$.

Alternatively, use $\frac{dV}{dx} = 12(x - 10)(x - 30)$.

When $x = 5$, $\frac{dV}{dx} = 12(5)^2 - 480(5) + 3600 = 1500$.

That is, $\frac{dV}{dx} > 0$.

When $x = 15$, $\frac{dV}{dx} = 12(15)^2 - 480(15) + 3600 = -900$.

That is, $\frac{dV}{dx} < 0$.

This means that $x = 10$ corresponds to a maximum value of V.
Look at the sign of $\frac{dV}{dx}$ on both sides of $x = 30$.

When $x = 25$, $\frac{dV}{dx} = 12(25)^2 - 480(25) + 3600 = -900$.

That is, $\frac{dV}{dx} < 0$.

When $x = 35$, $\frac{dV}{dx} = 12(35)^2 - 480(35) + 3600 = 1500$.

That is, $\frac{dV}{dx} > 0$.

This means that $x = 30$ corresponds to a minimum value of x. In fact, the volume of the folded box is zero when $x = 30$ because cutting squares of this size from each corner of the original sheet would leave no cardboard at all to fold into a box.

c From **b**, the maximum volume of the folded box occurs when $x = 10\,\text{cm}$.

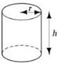
Then $V = 4(10)^3 - 240(10)^2 + 3600(10) = 16\,000\,\text{cm}^3$

The optimum dimens of the box are length = width = 40 and height = 10 cm.

In many optimisation problems, the function to be maximised or minimised is dependent on two variables. However, usually one of these variables can be eliminated using additional information given about the situation.

Example 5

A manufacturer wishes to make cylindrical steel cans with a capacity of 500 ml using the smallest quantity of metal possible. (Remember that 1 millilitre $\equiv 1\,\text{cm}^3$.)



It is assumed that th can has no metal lip around the top or bottom, which woul have to be accountec by adding an extra te to the expression for

a Find an expression for the total surface area $S\,\text{cm}^2$ of a cylindrical can, in terms of its radius r cm only.

b Find, correct to one decimal place, the values of r and h, the can's height, that would give the smallest surface area.

Solution

a The total surface area of the can, S, is given by

$$S = 2\pi r^2 + 2\pi rh \qquad [1]$$

At the moment S is expressed as an equation with two variables, r and h. We are required to express it in terms of r only, so we must eliminate h.
The volume of the cylinder is given by $V = \pi r^2 h$, where h is the height of the can.
For this particular can the volume is $500\,\text{cm}^3$, so

$$\pi r^2 h = 500 \qquad [2]$$

Rearranging equation [2], $h = \dfrac{500}{\pi r^2}$.

Substituting this expression for h into equation [1],

$$S = 2\pi r^2 + 2\pi r \frac{500}{\pi r^2}$$
$$= 2\pi r^2 + \frac{1000}{r}$$

Eliminating πr in the second term.

b $$\frac{dS}{dr} = 4\pi r - 1000r^{-2}$$
$$= 4\pi r - \frac{1000}{r^2}$$

At the maximum and minimum values of surface area, $\frac{dS}{dr} = 0$.

$$\frac{dS}{dr} = 0 \Rightarrow 4\pi r - \frac{1000}{r^2} = 0$$
$$\Rightarrow 4\pi r^3 - 1000 = 0$$
$$\Rightarrow \quad r^3 = \frac{1000}{4\pi} = \frac{250}{\pi}$$
$$\Rightarrow \quad r = \sqrt[3]{\frac{250}{\pi}} = 4.3 \text{ cm (1 d.p.)}$$

Multiply throughout by r^2.

Substituting for r in the rearranged form of equation [2] to find the corresponding height,

$$h = \frac{500}{\pi r^2} = \frac{500}{\pi} \times \frac{1}{\left(\sqrt[3]{\frac{250}{\pi}}\right)^2}$$
$$= \frac{500}{\pi} \times \left(\frac{250}{\pi}\right)^{-2/3}$$
$$= \frac{500}{\pi} \times \left(\frac{\pi}{250}\right)^{2/3}$$
$$= 8.6 \text{ cm (1 d.p.)}$$

Use the surd form of r, to avoid introducing inaccuracies due to rounding.

To decide whether these values for the radius and height correspond to a maximum or minimum value for the surface area, look at the value of $\frac{dS}{dr}$ on either side of $r = 4.3$.

When $r = 4$, $\frac{dS}{dr} = 16\pi - \frac{1000}{16} \approx -12.2$. That is, $\frac{dS}{dr} < 0$.

When $r = 5$, $\frac{dS}{dr} = 20\pi - \frac{1000}{25} \approx 22.8$. That is, $\frac{dS}{dr} > 0$.

This means that the surface area of a 500 ml cylindrical can is minimised when $r = 4.3$ cm and $h = 8.6$ cm.

These values of r and h minimise the surface area. The manufacturer may be more interested in minimising wasted metal when cutting the pieces, and this may give different values of r and h.

5.2 Stationary Points

Exercise

Technique

1 For each of the following quadratic functions:

i find the coordinates of the points where the graph of $y = f(x)$ crosses the axes

ii find the coordinates of the stationary point on the graph, and determine its nature

iii sketch the graph of $y = f(x)$. Check your graph using a graphical calculator.

a $f(x) = x^2 - 2x - 8$ b $f(x) = 35 + 2x - x^2$

c $f(x) = 4x^2 - 16x - 9$ d $f(x) = x^2 + 8x + 28$

2 For each of the following cubic functions, find the coordinates of any stationary points, and determine their nature:

a $y = x^3 + 6x^2 + 12x + 7$ b $y = 2x^3 - 12x^2 + 18x - 5$

c $y = x^3 + 7x^2 + 19x + 2$ d $y = 3x^3 - 3x^2 + 1$

3 a Find the coordinates of the points where the quartic curve $y = x^4 - 12x^3$ crosses the axes.

b Find $\frac{dy}{dx}$. Hence find the location, and determine the nature, of the stationary points on the curve.

c Sketch $y = x^4 - 12x^3$. Check your graph using a graphical calculator.

4 For each of the following functions, find an expression for $f'(x)$, and hence locate and determine the nature of any stationary points on the graph of $y = f(x)$:

a $f(x) = x + \dfrac{4}{x}$ b $f(x) = \dfrac{50}{x} - \dfrac{x^2}{5}$

c $f(x) = \dfrac{8}{x^2} + \dfrac{x}{4}$ d $f(x) = \dfrac{3}{x} - \dfrac{9}{x^2}$

5 Given that $s = 3t^2 - 8t + 3$, find the minimum value of s and the value of t for which this occurs. 5

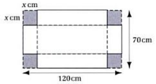
6 Given that $v = 26 + 11r - r^2$, find the maximum value of v and the value of r for which this occurs.

7 Find the values of x for which $f(x) = 20 + 8x - x^2$ is a decreasing function of x. 7

8 Find the values of t for which $g(t) = t^3 + 3t^2 - 9t + 6$ is an increasing function of t.

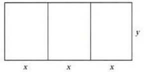
Contextual

1 An open box is formed from a 120 cm × 70 cm rectangular sheet of cardboard by cutting squares of side length x cm from each corner. The remaining flaps are then folded up to make the vertical sides of the box.



- **a** Find an expression in terms of x for the volume V (in cm^3) of the box.
- **b** Find $\frac{dV}{dx}$ and then solve the equation $\frac{dV}{dx} = 0$.
- **c** Find the maximum possible volume of the box. Justify your answer.

2 A garden centre wishes to use fencing to enclose three equally sized rectangular plots next to each other, as shown in the diagram. The total area A of the three plots is to be $288\ \text{m}^2$.



- **a** Find expressions for the total area A (in m^2), and the total length L (in m) of fencing required in terms of x and y.
- **b** Express L in terms of x only.
- **c** Hence, find the dimensions of each rectangular plot if the total length of fencing is to be kept as low as possible. Justify your answer.
- **d** What is the minimum length of fencing required?

3 On a particular day, the *Financial Times* 100 Share Index (FTSE) opens in London at 4000. During the rest of the day, its value t hours after the start of trading at 9 a.m. is given by $F = 4000 - 16t^2 + 8t^3 - \frac{3}{4}t^4$. A broker is instructed to sell her client's shares only if the value of the FTSE is falling.

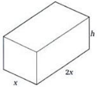
- **a** What is the value of the FTSE at noon?
- **b** Calculate the highest value of the index during the day. To the nearest minute, at what time does this occur?

c If trading finishes at 4.30 p.m., by how much has the index risen or fallen during the day?

d During which times of the day could the broker have sold her clients shares?

4 A cuboidal water tank, of height h cm, width x cm and length $2x$ cm, is designed to hold 700 litres when full.

1 litre $\equiv 1000\,\text{cm}^3$.



a Show that $h = \frac{350\,000}{x^2}$.

b Find an expression for the total surface area, $S\,\text{m}^2$, of the six faces of the tank in terms of x only.

c Find the dimensions of the tank that correspond to the least surface area. Justify your answer. (Give your answers to the nearest cm.)

5 A school decides to organise a monthly raffle in order to raise funds. It estimates that 2000 tickets would be bought if the price of each one was 50 pence, and that only 1000 tickets would be bought if they cost £1 each. The cost c of organising the raffle each month is £150 for prizes and 2 pence per ticket for printing.

a The number of tickets sold is modelled by $n = a + bs$, where s pence is the selling price, and a and b are constants. Find a and b.

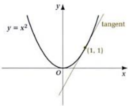
b Show that $c = (21\,000 - 40s)$ pence.

c Show that, in terms of the selling price s, the monthly profit generated by the raffle is given by $p = 3040s - 21\,000 - 20s^2$, and find the selling price that would maximise the profit. Calculate the maximum profit and the number of tickets sold to achieve it.

5.3 Further Applications of Differentiation

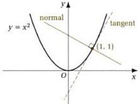
Tangents and normals

The gradient of a tangent drawn to a curve at any particular point is the same as the gradient of the curve at this point. This means that the gradient of a tangent can be found by differentiation. For example, consider the tangent drawn to the curve $y = x^2$ at the point $(1, 1)$.



The gradient function of this curve is $\frac{dy}{dx} = 2x$. So the gradient of the curve, and therefore the gradient of the tangent, at the point $(1, 1)$ is 2. Recall from Chapter 2 that the equation of a line with gradient m, that passes through a point with coordinates (x_1, y_1), is given by $\boldsymbol{y - y_1 = m(x - x_1)}$ So the equation of this particular tangent is given by $y - 1 = 2(x - 1)$ or $y = 2x - 1$.

The normal to a curve at any point is the straight line that passes through the curve at right-angles to the tangent at that point. Because the tangent and normal are perpendicular to each other, their gradients, m_1 and m_2 respectively, satisfy the condition $m_1 m_2 = -1$. So,



$$\text{gradient of normal} = \frac{-1}{\text{gradient of tangent}}$$ ◀ **Learn this result.**

So the gradient of the normal to the curve $y = x^2$ at $(1, 1)$ is $-\frac{1}{2}$. Now use $y - y_1 = m(x - x_1)$ to find the equation of this normal.

$$y - 1 = -\tfrac{1}{2}(x - 1)$$
$$\Rightarrow y - 1 = -\tfrac{1}{2}x + \tfrac{1}{2}$$
$$\Rightarrow \quad y = -\tfrac{1}{2}x + \tfrac{3}{2}$$

Example 1

Find the gradient of the curve $y = (x + 2)(4 - x)$ at the point where $x = 3$. Hence, find the equations of the tangent and the normal to the curve at this point.

Solution

$$y = (x + 2)(4 - x)$$
$$= 8 + 2x - x^2$$

So $\dfrac{dy}{dx} = 2 - 2x$

When $x = 3$, $y = 5$ and $\frac{dy}{dx} = -4$.

So the gradient of the tangent to the curve at $(3, 5)$ is -4.

This means that the gradient of the normal is $\frac{1}{4}$.

Now use $y - y_1 = m(x - x_1)$ to find the equations of both the tangent and the normal to the curve at $(3, 5)$.

The equation of the tangent is $y - 5 = -4(x - 3)$

$\Rightarrow \quad y - 5 = -4x + 12$

$\Rightarrow \quad y = -4x + 17$

The equation of the normal is $y - 5 = \frac{1}{4}(x - 3)$

$\Rightarrow \quad y = \frac{1}{4}x + \frac{17}{4}$

$\Rightarrow \quad x - 4y + 17 = 0$

Recall that straight line equations, involving fractions, can be rearranged into the form $ax + by + c = 0$.

Use a graphical calculator or graph plotting software to draw the quadratic curve $y = (x + 2)(4 - x)$. On the same graph plot the lines $y = -4x + 17$ and $y = \frac{1}{4}x + \frac{17}{4}$. Verify that they are the tangent and normal to the curve at the point $x = 3$.

Graphic
calculat
support
pack

Ensure that you have 'square' view; that is, one in which the x- a
y-axes have the same scale.

Example 2

Find the equation of the normal to the curve $y = x^2 + 4x - 2$ at the point where $x = -3$. Find the coordinates of the other point where this normal intersects $y = x^2 + 4x - 2$.

Solution

$$y = x^2 + 4x - 2$$

So $\dfrac{dy}{dx} = 2x + 4$

When $x = -3$, $y = -5$ and $\frac{dy}{dx} = -2$.
Since the gradient of the tangent to the curve at $(-3, -5)$ is -2, the gradient of the normal will be $\frac{1}{2}$.
Use $y - y_1 = m(x - x_1)$ to find the equation of this normal.

$$y - (-5) = \tfrac{1}{2}(x - (-3))$$
$$\Rightarrow \quad y + 5 = \tfrac{1}{2}x + \tfrac{3}{2}$$
$$\Rightarrow \quad y = \tfrac{1}{2}x + \tfrac{3}{2} - 5$$
$$\Rightarrow \quad y = \tfrac{1}{2}x - \tfrac{7}{2}$$

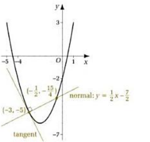
At the points of intersection of this normal and the curve $y = x^2 + 4x - 2$,

$$x^2 + 4x - 2 = \tfrac{1}{2}x - \tfrac{7}{2}$$
$$\Rightarrow \quad 2x^2 + 8x - 4 = x - 7$$
$$\Rightarrow \quad 2x^2 + 7x + 3 = 0$$
$$\Rightarrow \quad (2x + 1)(x + 3) = 0$$
$$\Rightarrow \quad x = -\tfrac{1}{2}, \text{ or } x = -3$$

Recall factorisation using PAFF: $P = 6$, $A = 7$, $F = 6, 1$.

This means the x-coordinate of the other point of intersection is $-\frac{1}{2}$. The corresponding y-coordinate can be found by substituting $x = -\frac{1}{2}$ into the equation of the normal. Check that this gives $y = -\frac{15}{4}$. So the normal to the curve at $(-3, -5)$ crosses the curve again at $(-\frac{1}{2}, -\frac{15}{4})$. Check this result using a graphical calculator and the TRACE facility.

The y-coordinate could also be found by substituting this value of x into the equation for the curve.



Displacement, velocity and acceleration

The branch of mathematics concerned with the study of moving objects, and in particular their displacement, velocity and acceleration, is called **kinematics**.

The **displacement** s of an object moving in a straight line is the distance it has travelled from a fixed point on the line in a specified direction. In many real-life situations, displacement can be a function of time t.

The **velocity** v of an object is the speed with which it is travelling in the specified direction. By definition, velocity is the rate at which the object's displacement is changing with respect to time. So $v = \frac{ds}{dt}$.

If displacement is measured in metres and time is measured in seconds, then the units of velocity are metres per second (written m/s or $\mathrm{m\,s^{-1}}$).

- If $v = 0$ then the object is stationary.
- If $v > 0$ then the object is moving along the line in the specified direction.
- If $v < 0$, then the object is moving in the opposite direction.

The **acceleration** a of a moving object is the rate at which its speed in the specified direction is changing. So acceleration is the rate of change of velocity with respect to time:

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

Notice that acceleration is the first derivative of velocity, and the second derivative of displacement. If velocity is measured in $\mathrm{m\,s^{-1}}$ then the units of acceleration are metres per second per second, or 'metres per second squared' (written $\mathrm{m/s^2}$ or $\mathrm{m\,s^{-2}}$).

- If $a = 0$ the object is moving with constant velocity (that is at constant speed in a straight line).
- If $a > 0$ the object is **accelerating**; that is its speed in the specified direction is increasing.
- If $a < 0$, then the object is **decelerating**; that is its speed in the specified direction is decreasing.

Example 3

The height h metres at time t seconds of a ball thrown vertically upwards, from a fixed point O, with initial velocity $12\,\mathrm{m\,s^{-1}}$, is given by $h = 12t - 5t^2$.

a Find the greatest height reached by the ball.
b Find the acceleration of the ball.
c Find the height of the ball after 0.3 and 1.7 seconds.
d Find the distance travelled by the ball between these two times.
e Find the average speed of the ball during this time interval.

Solution

a Height h takes the place of displacement s in this problem.
The velocity of the ball at any time is given by

$$v = \frac{dh}{dt} = 12 - 10t \quad (\text{m s}^{-1})$$

At its maximum height, the velocity of the ball is zero.

Then $12 - 10t = 0 \Rightarrow t = 1.2$ seconds

When $t = 1.2$, $h = 12(1.2) - 5(1.2)^2 = 7.2$

So, the greatest height reached is 7.2 m.

You must always include a unit in your answer if units are given in the question.

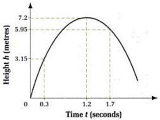
b Acceleration, $a = \frac{dv}{dt} = \frac{d}{dt}(12 - 10t) = -10\ (\text{m s}^{-2})$

This is the downwards acceleration due to gravity. A more accurate value would be $9.81\,\text{m s}^{-2}$.

c When $t = 0.3$, $h = 12(0.3) - 5(0.3)^2 = 3.15\,\text{m}$

When $t = 1.7$, $h = 12(1.7) - 5(1.7)^2 = 5.95\,\text{m}$

d Consider the graph of h against t.



The ball reaches its greatest height and changes direction between $t = 0.3$ seconds and $t = 1.7$ seconds. The distance travelled during this time interval is the sum of the distance the ball rises and the distance the ball then falls.
Between $t = 0.3$ seconds and $t = 1.2$ seconds, the ball rises and travels $(7.2 - 3.15) = 4.05$ metres upwards.
Between $t = 1.2$ seconds and $t = 1.7$ seconds, the ball falls and travels $(7.2 - 5.95) = 1.25$ metres downwards.
So it travels a total distance of $(4.05 + 1.25) = 5.3\,\text{m}$.

e
$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}} = \frac{5.3\,(\text{m})}{1.4\,(\text{s})} = 3.8\,\text{m s}^{-1}\ (1\text{ d.p.})$$

Small changes

If $y = f(x)$, then recall that by definition the gradient of its graph at any particular point is given by

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{limit}} \left(\frac{\delta y}{\delta x} \right)$$

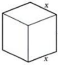
Remember that δy is the small change in the value of the y-coordinate at this point corresponding to a small change, δx, in the x-coordinate. Provided δx is very small,

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

This means that the ratio $\frac{\delta y}{\delta x}$ gives a good approximation to $\frac{dy}{dx}$, the gradient function at that point. Remember that $\frac{dy}{dx}$ is the gradient of the curve $y = f(x)$ at the point from which this small change is being made. This approximation allows the change in the value of y resulting from a small change in the value of x (or vice versa) to be calculated quickly.

Example 4

The side length of a 100 cm × 100 cm × 100 cm cube is increased by 2 cm. Find the approximate increase in the cube's surface area.



Solution

If the side length of the cube is x, then the surface area $S = 6x^2$.

$$\text{So } \frac{dS}{dx} = 12x.$$

Because the change in the side length, δx, is small relative to the original length,

$$\frac{\delta S}{\delta x} \approx \frac{dS}{dx}$$

where δS is the corresponding small change in surface area, and $\frac{dS}{dx}$ is the value of the gradient function when $x = 100$ cm. Therefore

$$\begin{aligned}\delta S &\approx \frac{dS}{dx} \times \delta x \\ &= 12x \times \delta x \\ &= 12(100) \times 2 = 2400\,\text{cm}^2\end{aligned}$$

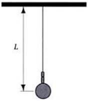
The approximate increase in the cube's surface area is $2400\,\text{cm}^2$.

To see how accurate this approximation is put $x = 102$ cm in the equation for surface area, $S = 6x^2$. The ne surface area is $62\,424\,\text{cm}^2$. The actu increase in surface ar is therefore $2424\,\text{cm}^2$ the approximation is good.

The change in a measurement or quantity is often expressed as a percentage of the original value. If $y = f(x)$, then $\pm\frac{\delta x}{x} \times 100$ is the percentage increase (+) or decrease (−) in the value of x, and $\pm\frac{\delta y}{y} \times 100$ is the corresponding percentage increase (+) or decrease (−) in the value of y.

Example 5

The time period T of a simple swinging pendulum is a function of the length L of the pendulum, such that $T = 2\pi\sqrt{L/g}$, where g is a constant. Find the percentage change in the period if the pendulum is shortened by 6%.



Solution

$$T = 2\pi\sqrt{\frac{L}{g}} = \frac{2\pi}{\sqrt{g}}L^{1/2}$$

So $$\frac{\mathrm{d}T}{\mathrm{d}L} = \frac{2\pi}{\sqrt{g}} \times \frac{1}{2}L^{-1/2} = \frac{\pi}{\sqrt{g}} \times \frac{1}{\sqrt{L}} = \frac{\pi}{\sqrt{gL}}$$

The percentage change in the value of L is −6%, the minus sign indicating a decrease. Therefore

$$\frac{\delta L}{L} = -0.06$$ ◀ $6\% \equiv \frac{6}{100} = 0.06$

Because this is a small change

$$\frac{\delta T}{\delta L} \approx \frac{\mathrm{d}T}{\mathrm{d}L}$$

$$\Rightarrow \frac{\delta T}{\delta L} \approx \frac{\pi}{\sqrt{gL}}$$ ◀ Using $\frac{\mathrm{d}T}{\mathrm{d}L} = \frac{\pi}{\sqrt{gL}}$.

$$\Rightarrow \delta T \approx \frac{\pi}{\sqrt{gL}}\delta L$$

$$\Rightarrow \frac{\delta T}{T} \approx \frac{\pi}{\sqrt{gL}}\frac{\delta L}{T}$$

$$= \frac{\pi}{\sqrt{gL}} \times \frac{1}{2\pi}\sqrt{\frac{g}{L}} \times \delta L$$ ◀ Using $T = 2\pi\sqrt{\frac{L}{g}}$.

That is, $\frac{\delta T}{T} \approx \frac{1}{2} \times \frac{\delta L}{L} = \frac{1}{2} \times -0.06 = -0.03$.

This means, the percentage change in the period of the pendulum, as the length is shortened, is a decrease of 3%.

The minus sign indicates a decrease in the value of T.

5.3 Further Applications of Differentiation

Exercise

Technique

1 Find the equations of the tangent and normal to each of the following curves at the points indicated:

Give your answers in the form $ax + by + c = 0$.

1 a

- **a** $y = x^2 - 5x + 1$ at $(6, 7)$
- **b** $y = (2x + 1)(x - 5)$ at $(1, -12)$
- **c** $y = x^3 - 6x^2 + 3x + 1$ at $(3, -17)$
- **d** $y = x + \frac{3}{x}$ at $(2, 3\frac{1}{2})$
- **e** $y = \sqrt{x}$ at $(9, 3)$
- **f** $y = 2x - x^3$ at $(-\frac{1}{2}, -\frac{7}{8})$

2 Find the equations of the tangents to the curve $y = x^2 - 4x + 3$ at the points where it crosses the x-axis. Show that these tangents intersect at the point $(2, -2)$.

3 Show that the equation of the tangent to the curve $y = x^3 - 6x^2 + 3x + 10$ at the point where it crosses the y-axis is $y = 3x + 10$. Find the coordinates of the other point on the curve whose tangent is parallel to $y = 3x + 10$. Find the equation of this second tangent.

4 Find the equation of the normal to the curve $y = x - \frac{6}{x}$ at the point where $x = 3$. Find the coordinates of the other point where this normal intersects $y = x - \frac{6}{x}$.

Contextual

1 The height, h metres, of an object thrown vertically upwards at time t seconds after its release is given by $h = 40t - 5t^2$.

- **a** Calculate how long it takes for the object to return to its point of projection.
- **b** Find the value of t at which the object is momentarily stationary, and hence calculate the maximum height reached by the object.

2 The displacement, s metres, of an object from some fixed point O at time t seconds is given by $s = t^3 - 3t^2 + 4t + 5$.

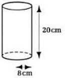
- **a** Find expressions for the velocity v and acceleration a at time t.
- **b** Show that the object is never stationary.
- **c** Calculate the average speed of the object during the first three seconds.

3 A closed cylinder has a base radius of 8 cm and a height of 20 cm. Calculate the small changes

a in its volume and

b in it surface area

that result from a small change of 0.1 cm in its radius. Leave your answers in terms of π and assume height remains constant.



4 A cuboidal box has height x, length $3x$, and width $2x$. Calculate the percentage increase in the value of x if the volume is to increase by 4.5%.

5.4 The Chain Rule and Related Rates of Change

Functions such as $y = (x+1)^3$ and $y = (2x-3)^5$ are examples of composite functions. They are also known as 'functions of a function'. Notice that $y = (x+1)^3$ can be written as $y = u^3$ where $u = x+1$.

One method of finding derivatives of composite functions is to expand the brackets and then differentiate term-by-term. For example,

$$\begin{aligned} y &= (x+1)^3 \\ &= (x+1)(x+1)(x+1) \\ &= (x^2+2x+1)(x+1) \\ &= x^3+3x^2+3x+1 \end{aligned}$$

$$\begin{aligned} \text{So } \frac{dy}{dx} &= 3x^2+6x+3 \\ &= 3(x^2+2x+1) \\ &= 3(x+1)^2 \end{aligned}$$

Notice that the gradie function includes the bracket featured in th original function, rais to a lower power.

Try finding $\frac{d}{dx}(2x-3)^5$ in the same way. What happens when you differentiate term-by-term? Notice that the algebra gets quite complicated. Factorisation of such expressions, which we would need to do to locate stationary points, can be very tricky. This is a major disadvantage of this 'expansion and term-by-term differentiation' method.

We also need an alternative method of differentiating composite functions such as $y = (4-x)^{-1}$ and $y = \sqrt{3x+1}$, which cannot be expanded into a finite number of terms involving powers.

Differentiating composite functions – the chain rule

Suppose $y = f(x)$ is a composite function of x. This means that $y = f(u)$, where $u(x)$ is some intermediate function of x that can be identified in the construction of function f. Any small change in the value of x, δx, gives rise to a small change in the value of u, δu. This then gives rise to a small change in the value of y, δy. Differentiating from first principles,

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$$

but since it is possible to write

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

it follows that

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{limit}} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} \right)$$

However, $\delta u \to 0$ as $\delta x \to 0$, and therefore

$$\frac{dy}{dx} = \underset{\delta u \to 0}{\text{limit}} \left(\frac{\delta y}{\delta u} \right) \times \underset{\delta x \to 0}{\text{limit}} \left(\frac{\delta u}{\delta x} \right)$$

This gives

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

◀ **Learn this important result.**

This method of differentiating composite functions by introducing an intermediate variable u is the **chain rule.**

Example 1

Differentiate the following with respect to x, using the chain rule.

a $y = (x + 1)^3$ **b** $y = (2x - 3)^5$

Solution

a For $y = (x + 1)^3$, let $y = u^3$, where $u = x + 1$.

Therefore $\dfrac{dy}{du} = 3u^2$ and $\dfrac{du}{dx} = 1$.

Differentiating y with respect to u and u with respect to x.

Using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 \times 1 = 3u^2$$

Substituting $u = x + 1$, $\dfrac{dy}{dx} = 3(x + 1)^2$

b For $y = (2x - 3)^5$, let $y = u^5$ where $u = 2x - 3$.

$$\frac{dy}{du} = 5u^4 \quad \text{and} \quad \frac{du}{dx} = 2$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5u^4 \times 2 = 10u^4$$

Substituting $u = 2x - 3$, $\dfrac{dy}{dx} = 10(2x - 3)^4$

Example 2

Use the chain rule to find the gradient function for each of the following:

a $y = \dfrac{1}{4x - 5}$ **b** $y = \sqrt{3x - 1}$

Solution

a Before using the chain rule, $y = \dfrac{1}{4x-5}$ must be written as a bracket raised to a power. In this case, $y = \dfrac{1}{4x-5} = (4x-5)^{-1}$.
Now let $y = u^{-1}$, where $u = 4x - 5$.

Then $\dfrac{dy}{du} = -u^{-2} = -\dfrac{1}{u^2}$ and $\dfrac{du}{dx} = 4$.

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{u^2} \times 4 = -\frac{4}{u^2}$$

Substituting $u = 4x - 5$, $\dfrac{dy}{dx} = -\dfrac{4}{(4x-5)^2}$

b Similarly, write $y = \sqrt{3x-1}$ as $y = (3x-1)^{1/2}$.
Now let $y = u^{1/2}$, where $u = 3x - 1$.

Then $\dfrac{dy}{du} = \dfrac{1}{2}u^{-1/2} = \dfrac{1}{2\sqrt{u}}$ and $\dfrac{du}{dx} = 3$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{u}} \times 3 = \frac{3}{2\sqrt{u}}$$

Substituting $u = 3x - 1$, $\dfrac{dy}{dx} = \dfrac{3}{2\sqrt{3x-1}}$

When using the chain rule, the choice of the intermediate function $u(x)$ is very important. It is essential that $u(x)$ can be easily differentiated.

In general, any composite function of the form $y = [f(x)]^n$, involving some function $f(x)$ raised to a rational power n, can be differentiated using the chain rule.

Let $y = u^n$, where $u = f(x)$.

Then $\dfrac{dy}{du} = nu^{n-1}$, and $\dfrac{du}{dx} = f'(x)$.

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

So $\dfrac{dy}{dx} = nu^{n-1} \times f'(x)$

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \times f'(x)$$

Note that $n[f(x)]^{n-1}$ is the derivative of the bracket ignoring its contents, and $f'(x)$ is the derivative of the contents of the bracket.

Example 3

The tangent to the curve $y = \dfrac{5}{1+x^2}$ at $x = 2$ crosses the y-axis at A and the x-axis at B.

a Find the equation of the tangent.
b Find the coordinates of points A and B.
c Show that the area of triangle OAB is $\frac{169}{40}$ square units.

Solution

a In order to find the equation of the tangent to the curve at $x = 2$, first calculate the gradient of the curve at this point.

Use the chain rule to differentiate $y = \dfrac{5}{1+x^2}$.

Let $y = \dfrac{5}{u} = 5u^{-1}$, where $u = 1 + x^2$.

Then $\dfrac{dy}{du} = -5u^{-2}$, and $\dfrac{du}{dx} = 2x$.

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{5}{u^2} \times 2x$$

Substituting $u = 1 + x^2$, $\dfrac{dy}{dx} = -\dfrac{10x}{(1+x^2)^2}$

Now, when $x = 2$,

$$\frac{dy}{dx} = -\frac{10 \times 2}{(1+2^2)^2} = -\frac{20}{25} = -\frac{4}{5}$$

The gradient of the curve, and therefore the gradient of the tangent drawn to the curve, at $x = 2$ is $-\frac{4}{5}$.
Use $y - y_1 = m(x - x_1)$ to find the equation of the tangent.

When $x = 2$, $y = \dfrac{5}{1+2^2} = 1$

So $\quad y - 1 = -\dfrac{4}{5}(x - 2)$

$$y = -\frac{4}{5}x + \frac{13}{5}$$

or $5y + 4x - 13 = 0$

b At point A, where the tangent crosses the y-axis, $x = 0 \Rightarrow y = \frac{13}{5}$.
The coordinates of A are $(0, \frac{13}{5})$.

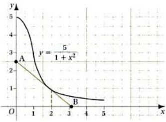
At point B, where the tangent crosses the x-axis,

$$y = 0 \Rightarrow -\frac{4}{5}x + \frac{13}{5} = 0$$

$$\Rightarrow \qquad x = \frac{13}{4}$$

The coordinates of B are $(\frac{13}{4}, 0)$

c



Recall that the area of a triangle is given by
area $= \frac{1}{2}$ base $\times$ perpendicular height.
So area of triangle OAB $= \frac{1}{2} \times \frac{13}{4} \times \frac{13}{5} = \frac{169}{40}$ square units.

Related rates of change

Consider how to calculate the rate at which the area of a circular oil slick, of uniform thickness, changes. Using the area of a circle, $A = \pi r^2$, the rate of change of area with respect to the slick's radius r, $\frac{dA}{dr}$ can be found by differentiation. That is, $\frac{dA}{dr} = 2\pi r$.

From aerial observation, or by more detailed mathematical modelling, it may be possible to determine the rate of change of the radius with respect to time, $\frac{dr}{dt}$.

The chain rule can now be used to link these two related rates of change together. The rate of change of the slick's area with respect to time is given by

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

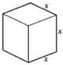
$$= 2\pi r \frac{dr}{dt}$$

It is important to remember that if a particular variable is increasing then its rate of change will be positive, and if it is decreasing, its rate of change will be negative.

Unless specified otherwise, the phrase 'rate of change' refers to the rate at which a variable is changing *with respect to time*.

Example 4

A metallic cube, of side length x cm, is being heated in a furnace. The side lengths are expanding at the rate of $0.1\,\text{cm}\,\text{s}^{-1}$. Find the rates at which the cube's surface area and the cube's volume are changing when $x = 10$ cm.



Solution

The surface area of the cube, $S = 6x^2$. Hence $\frac{dS}{dx} = 12x$.

The volume of the cube, $V = x^3$. So $\frac{dV}{dx} = 3x^2$.

Since the side lengths are expanding at the rate of $0.1\,\text{cm}\,\text{s}^{-1}$,

$$\frac{dx}{dt} = 0.1$$

Using the chain rule, the rate of change of surface area,

$$\frac{dS}{dt} = \frac{dS}{dx} \times \frac{dx}{dt}$$
$$= 12x \times 0.1 = 1.2x\,\text{cm}^2\,\text{s}^{-1}$$

When $x = 10$ cm, $\frac{dS}{dt} = 12\,\text{cm}^2\,\text{s}^{-1}$.

Using the chain rule, the rate of change of volume,

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$
$$= 3x^2 \times 0.1 = 0.3x^2\,\text{cm}^3\,\text{s}^{-1}$$

When $x = 10$ cm, $\frac{dV}{dt} = 30\,\text{cm}^3\,\text{s}^{-1}$.

Example 5

Air is being pumped into a spherical balloon at the rate of $300\,\text{cm}^3\,\text{s}^{-1}$. Find the rate at which the surface area of the balloon is increasing when the radius is 15 cm.

Solution

The rate of change of volume, $\frac{dV}{dt} = 300\,\text{cm}^3\,\text{s}^{-1}$.
Since the surface area of a sphere is $S = 4\pi r^2$, its rate of change with respect to radius r is $\frac{dS}{dr} = 8\pi r$.
But it is not possible to link these two related rates of change together to form an expression for $\frac{dS}{dt}$, the rate at which the surface area is changing with respect to time. Instead, we must involve a third expression.

Using the chain rule,

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$$

Since the volume of this spherical balloon is $V = \frac{4}{3}\pi r^3$, then

$$\frac{dV}{dr} = 4\pi r^2$$

and so $\frac{dr}{dV} = \frac{1}{4\pi r^2}$

$$\frac{dr}{dV} = \frac{1}{\left(\frac{dV}{dr}\right)}$$

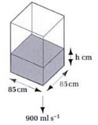
The rate of change of surface area is then

$$\begin{aligned}\frac{dS}{dt} &= \frac{dS}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt} \\ &= 8\pi r \times \frac{1}{4\pi r^2} \times 300 \\ &= \frac{600}{r}\ \text{cm}^2\,\text{s}^{-1}\end{aligned}$$

When $r = 15$ cm, $\frac{dS}{dt} = 40\ \text{cm}^2\,\text{s}^{-1}$.

Example 6

Water is emptying out of a 85 cm × 85 cm square-based cuboidal tank at the rate of 900 millilitres per second. Find, correct to 2 decimal places, the rate at which the height of the water is falling in the tank. Calculate how long it takes for the height to fall by 10 cm.



1 millilitre ≡ 1 cm³.

Solution

$$\begin{aligned}\frac{dV}{dt} &= -900\ \text{ml}\,\text{s}^{-1} \\ &= -900\ \text{cm}^3\,\text{s}^{-1}\end{aligned}$$

The minus sign indicates that the volume is decreasing.

The volume of the water left in the tank, when its height is h cm, is given by

$$\begin{aligned}V &= 85 \times 85 \times h \\ &= 7225h\ \text{cm}^3\end{aligned}$$

This means the rate of change of volume V with respect to height h is $\frac{dV}{dh} = 7225$.

Using the chain rule, the rate of change of height,

$$\begin{aligned}\frac{dh}{dt} &= \frac{dh}{dV} \times \frac{dV}{dt} \\ &= \frac{1}{7225} \times -900 \\ &= -0.125\,\text{cm}\,\text{s}^{-1} \quad \text{(3 d.p.)}\end{aligned}$$

$$\frac{dh}{dV} = \frac{1}{\left(\frac{dV}{dh}\right)}$$

This means the height falls by approximately 0.125 cm per second.
The fall in the height of the water in the tank over a period of time can be found using

fall in height of water = fall per second × time taken

The time taken for the height of the water to fall 10 cm is given by

$$\begin{aligned}\text{time taken} &= \frac{\text{fall in height of water}}{\text{fall per second}} \\ &= \frac{10\,\text{cm}}{0.125\,\text{cm}\,\text{s}^{-1}} \\ &= 80 \text{ seconds (to nearest second)}\end{aligned}$$

5.4 The Chain Rule and Related Rates of Change

Exercise

Technique

1 Use the chain rule to find $\frac{dy}{dx}$ for each of the following:

a $y = (x+2)^4$

b $y = (3-x)^5$

c $y = (4x-5)^3$

d $y = (x^2+1)^6$

e $y = (2x^3 - x + 1)^5$

f $y = \dfrac{1}{5x+9}$

g $y = \sqrt{2x+3}$

h $y = (x^4+2)^{5/2}$

i $y = \dfrac{1}{(7-6x)^2}$ 1 d

j $y = \dfrac{1}{\sqrt{25-x^2}}$

2 Use the chain rule to find the derivative of each of the following functions:

a $s(t) = (2t+5)^7$

b $h(r) = (9r-4)^{-2}$

c $v(t) = \sqrt{t^2-2}$

d $A(\theta) = \sqrt[3]{6\theta+2}$

e $P(t) = \dfrac{3}{\sqrt{1-t}}$

f $V(x) = \dfrac{12}{x^3+2x}$

g $g(s) = s + \dfrac{1}{s}$

h $l(u) = (1+\sqrt{u})^3$

3 Find the gradient of each of the following curves at the point indicated:

a $y = (x-3)^5$ at $(5, 32)$

b $y = (2x+3)^4$ at $(-1, 1)$

c $y = (13-5x)^3$ at $(3, -8)$

d $y = \dfrac{5}{3x+1}$ at $(-2, -1)$ 3 d

e $y = \sqrt{4x^2+9}$ at $(2, 5)$.

4 Find the coordinates of the point(s) on each of the following curves at which the gradient has the value indicated:

a $y = (x+6)^4$ where $\dfrac{dy}{dx} = -4$

b $y = (2x-8)^3$ where $\dfrac{dy}{dx} = 24$

c $y = \sqrt{x-7}$ where $\dfrac{dy}{dx} = \dfrac{1}{6}$

5 Given that $p = s^2 + 3s - 4$ and $s = 5t + 1$, find $\frac{dp}{ds}$ and $\frac{ds}{dt}$.
Hence, use the chain rule to find an expression for $\frac{dp}{dt}$ in terms of t only.

6 Given that $V = (3y-2)^4$ and $y = (1+x)^2$, use the chain rule to find an expression for $\frac{dV}{dx}$ in terms of x only.

Contextual

1 The radius of a circular oil slick is increasing at the rate of $\dfrac{500}{100+r^2}$ metres per hour. Find the rates at which the slick's perimeter and area are changing when the radius is 20 metres (leave your answers in terms of π).

1

2 As air is pumped into a spherical balloon, the rate at which its surface area increases remains a constant $16\pi\,\text{cm}^2\,\text{s}^{-1}$.

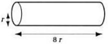
a Find the rate at which the radius of the balloon is changing when $r = 5$ cm.

b What is the volume of air in the balloon (in litres) when the rate at which the radius is increasing has dropped to $0.1\,\text{cm}\,\text{s}^{-1}$?

Remember, $1\text{ litre} \equiv 1000\,\text{cm}^3$

3 Find the coordinates of the stationary points on the graph of $y = x + \dfrac{1}{4x+6}$ and determine their nature.

4 A cylindrical metal rod, of radius r cm and length $8r$ cm, is being heated in a furnace. Its volume increases at the constant rate of $48\pi\,\text{cm}^3$ per second.

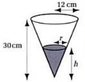


a Express the volume V in terms of r only. Hence, find $\frac{dV}{dr}$.

b Use the chain rule to calculate the rate at which the radius is increasing when $r = 5$ cm.

c Find an expression for the surface area S of the rod in terms of r. Hence, calculate the rate at which it is changing when $r = 5$ cm.

5 A right-circular cone, of height 30 cm and radius 12 cm, is filled with sand. The sand is then allowed to drain from the apex of the cone at the rate of $30\,\text{cm}^3$ per second. At any given time, it can be assumed that the remaining sand forms a right-circular cone of height h and radius r.



Recall that a right-circular cone is one with its apex directly above or below the centre of the circular base.

a Express the volume V of remaining sand in terms of h only. (Hint: The proportions of this particular cone are such that $h = \frac{5}{2}r$).

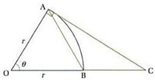
b Find an expression for $\frac{dV}{dh}$. Use the chain rule to calculate the rate at which the height of the remaining sand is falling when $h = 15$ cm.

c How long, to the nearest second, does it take for the initially full cone to empty?

5.5 Differentiation of Trigonometric Functions

Small angle approximations

In order to be able to differentiate the sine, cosine and tangent functions from first principles, their behaviour for *small* angles must first be established.



△ OAC is right-angle
so $\tan\theta = \dfrac{AC}{OA} = \dfrac{AC}{r}$.

In the diagram, AB is an arc of radius r which subtends an angle θ at its centre O. The line AC is a tangent to this arc at the point A. This means that triangle OAC is right-angled at A. Since $AC = r\tan\theta$, it follows that

$$\begin{aligned}\text{area of triangle OAC} &= \tfrac{1}{2}\text{ base } \times \text{perpendicular height}\\ &= \tfrac{1}{2}r \times r\tan\theta\\ &= \tfrac{1}{2}r^2\tan\theta\end{aligned}$$

Also, area of triangle OAB $= \frac{1}{2}r^2\sin\theta$.

Given that angle θ is measured in radians, area of sector OAB $= \frac{1}{2}r^2\theta$. From the diagram, notice that

area of triangle OAB < area of sector OAB < area of triangle OAC

Recall from Chapter the other formula for area of a triangle, area $= \frac{1}{2}ab\sin C$.

Recall from Chapter that sector area $= \frac{1}{2}r^2$ (when θ is measured radians).

This means $\frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta$

So $\sin\theta < \theta < \tan\theta$

Dividing this inequality throughout by $\sin\theta$ gives

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Recall that for small angles, $\cos\theta \approx 1$. This means that for small θ,

$$\frac{\theta}{\sin\theta} \approx 1$$

So if θ is small and measured in radians, $\sin\theta \approx \theta$.

Alternatively, dividing $\sin\theta < \theta < \tan\theta$ throughout by $\tan\theta$ gives

$$\frac{\sin\theta}{\tan\theta} < \frac{\theta}{\tan\theta} < 1$$

$$\cos\theta < \frac{\theta}{\tan\theta} < 1$$

Again, since $\cos\theta \approx 1$ for very small values of θ in radians, it follows that

$$\frac{\theta}{\tan\theta} \approx 1$$

This means that $\tan\theta \approx \theta$ for small angles measured in radians.

The behaviour of the cosine function for small angles can be established using the Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\Rightarrow \qquad \cos^2\theta = 1 - \sin^2\theta$$

$$\Rightarrow \qquad \cos\theta = (1 - \sin^2\theta)^{\frac{1}{2}}$$

For small values of θ, $\sin\theta \approx \theta$, so

$$\cos\theta \approx (1 - \theta^2)^{\frac{1}{2}}$$

The expression $(1 - \theta^2)^{\frac{1}{2}}$ can be expanded using the binomial expansion (see Chapter 8), to give

$$\cos\theta \approx 1 - \tfrac{1}{2}\theta^2$$

In summary, the behaviour of sine, cosine and tangent functions for small angles measured in radians are:

$\sin\theta \approx \theta$

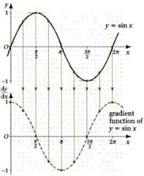
$\cos\theta \approx 1 - \frac{1}{2}\theta^2$

$\tan\theta \approx \theta$

◀ **Learn these important results.**

Differentiation of sine and cosine

The diagram shows the graphs of $y = \sin x$ and its gradient function for the interval $0 \leq x \leq 2\pi$. By considering the gradient of the sine curve at several values of x, it is possible to gain some insight into the nature of its gradient function.

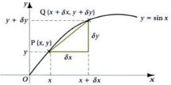


For example, at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, the gradient is zero. Between these two values, $\sin x$ is a decreasing function of x, and the gradient of the sine curve is therefore negative. The gradient is at its most negative when $x = \pi$, and this corresponds to a minimum point on the graph of the gradient function.

For $0 \leq x \leq \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$, $\sin x$ is an increasing function of x. The sine curve has a positive gradient in these intervals. The gradient function therefore reaches a maximum at $x = 0$ and $x = 2\pi$.

What do you notice about the graph of the gradient function? The outline graph of the gradient function obtained appears to resemble the curve $y = \cos x$. Its exact nature can be found algebraically by differentiating the sine function from first principles.

Consider the chord drawn from the point $P(x, y)$ to the point $Q(x + \delta x, y + \delta y)$ on the curve $y = \sin x$.



$$\text{gradient of the chord PQ} = \frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}} = \frac{\delta y}{\delta x}$$

$$= \frac{(y + \delta y) - y}{\delta x}$$

$$= \frac{\sin(x + \delta x) - \sin x}{\delta x}$$

Recall that if $y = \sin x$, then $y + \delta y = \sin(x + \delta x)$.

Now use the compound angle formula,

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

to give $\sin(x+\delta x) = \sin x \cos \delta x + \cos x \sin \delta x$.
This means

$$\frac{\delta y}{\delta x} = \left(\frac{\sin x \cos \delta x + \cos x \sin \delta x - \sin x}{\delta x}\right)$$

Differentiating from first principles, the gradient function for the curve $y = \sin x$ can be found by finding the limiting value of $\frac{\delta y}{\delta x}$ as $\delta x \to 0$.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)\\ &= \lim_{\delta x \to 0}\left(\frac{\sin x \cos \delta x + \cos x \sin \delta x - \sin x}{\delta x}\right)\end{aligned}$$

Now, use the small angle approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$.

For small values of δx, $\sin \delta x \approx \delta x$ and $\cos \delta x \approx 1 - \frac{1}{2}(\delta x)^2$.

Using these approximations,

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\delta x \to 0}\left(\frac{[1-\frac{1}{2}(\delta x)^2]\sin x + \delta x \cos x - \sin x}{\delta x}\right)\\ &= \lim_{\delta x \to 0}\left(\frac{\sin x - \frac{1}{2}(\delta x)^2 \sin x + \delta x \cos x - \sin x}{\delta x}\right)\\ &= \lim_{\delta x \to 0}\left(\cos x - \tfrac{1}{2}(\delta x)\sin x\right)\\ &= \cos x\end{aligned}$$

If $y = \sin x$ then $\frac{dy}{dx} = \cos x$ ◀ **Learn this result.**

Remember that this result is dependent on x being in radians.

We can also find the gradient function of the cosine function by differentiating from first principles.

If $y = \cos x$, then

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)\\ &= \lim_{\delta x \to 0}\left(\frac{\cos(x+\delta x) - \cos x}{\delta x}\right)\end{aligned}$$

Use the compound angle formula

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

to give $\cos(x + \delta x) = \cos x \cos \delta x - \sin x \sin \delta x$. Then

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{limit}} \left(\frac{\cos x \cos \delta x - \sin x \sin \delta x - \cos x}{\delta x} \right)$$

Use the small angle approximations, $\sin \delta x \to \delta x$ and $\cos \delta x \to 1 - \frac{1}{2}(\delta x)^2$ as $\delta x \to 0$,

$$\begin{aligned}\frac{dy}{dx} &= \underset{\delta x \to 0}{\text{limit}} \left(\frac{[1 - \frac{1}{2}(\delta x)^2] \cos x - \delta x \sin x - \cos x}{\delta x} \right) \\ &= \underset{\delta x \to 0}{\text{limit}} \left(\frac{\cos x - \frac{1}{2}(\delta x)^2 \cos x - \delta x \sin x - \cos x}{\delta x} \right) \\ &= \underset{\delta x \to 0}{\text{limit}} \left(-\tfrac{1}{2}(\delta x) \cos x - \sin x \right) \\ &= -\sin x\end{aligned}$$

If $y = \cos x$, then $\dfrac{dy}{dx} = -\sin x$

◀ **Learn this result.**

It is important to note that trigonometric functions can only be differentiated if angles are measured in radians. This is because the derivation of their gradient functions from first principles relies upon small angle approximations for $\sin \theta$ and $\cos \theta$. These are only valid for angles measured in radians.

The chain rule can be used to differentiate composite functions of sine or cosine involving double or multiple angles.

Example 1

Differentiate the following with respect to x:

a $y = \sin 2x$
b $y = 4 \cos 3x$
c $y = \sin(ax + b)$ where a and b are constants
d $y = \cos(x^2 + \pi)$.

Solution

a Let $y = \sin u$, where $u = 2x$.

Then $\dfrac{dy}{du} = \cos u$ and $\dfrac{du}{dx} = 2$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times 2$$

Substituting $u = 2x$, $\frac{dy}{dx} = 2\cos 2x$

b Let $y = 4\cos u$, where $u = 3x$.

Then $\frac{dy}{du} = -4\sin u$ and $\frac{du}{dx} = 3$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -4\sin u \times 3$$

Substituting $u = 3x$, $\frac{dy}{dx} = -12\sin 3x$

c Let $y = \sin u$, where $u = ax + b$.

Then $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = a$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times a$$

Substituting $u = ax + b$, $\frac{dy}{dx} = a\cos(ax + b)$

d Let $y = \cos u$, where $u = x^2 + \pi$.

Then $\frac{dy}{du} = -\sin u$ and $\frac{du}{dx} = 2x$

Using the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin u \times 2x$

Substituting $u = x^2 + \pi$, $\frac{dy}{dx} = -2x\sin(x^2 + \pi)$

The chain rule can also be used to differentiate functions involving powers of sine and cosine.

Example 2

Differentiate the following with respect to θ.

a $x = \sin^2 \theta$ **b** $y = \cos^3 4\theta$

Solution

a For $x = \sin^2\theta = (\sin\theta)^2$, let $x = u^2$, where $u = \sin\theta$.

Then $\dfrac{dx}{du} = 2u$ and $\dfrac{du}{d\theta} = \cos\theta$

Using the chain rule,

$$\frac{dx}{d\theta} = \frac{dx}{du} \times \frac{du}{d\theta} = 2u\cos\theta$$

Substituting $u = \sin\theta$, $\dfrac{dx}{d\theta} = 2\sin\theta\cos\theta = \sin 2\theta$

b For $y = \cos^3 4\theta = (\cos 4\theta)^3$, let $y = u^3$, where $u = \cos 4\theta$.

Then $\dfrac{dy}{du} = 3u^2$ and $\dfrac{du}{d\theta} = -4\sin 4\theta$

Use the chain rule to find $\frac{dy}{d\theta}$.

Using the chain rule,

$$\frac{dy}{d\theta} = \frac{dy}{du} \times \frac{du}{d\theta} = 3u^2 \times -4\sin 4\theta$$

Substituting $u = \cos 4\theta$, $\dfrac{dy}{d\theta} = -12(\cos 4\theta)^2 \sin 4\theta = -12\cos^2 4\theta \sin 4\theta$

Example 3

Find the equations of the tangent and normal to the curve $y = \cos 2x$ at the point where $x = \frac{\pi}{6}$.

Solution

If $y = \cos 2x$, then $\frac{dy}{dx} = -2\sin 2x$

When $x = \dfrac{\pi}{6}, y = \cos\dfrac{\pi}{3} = \dfrac{1}{2}$ and $\dfrac{dy}{dx} = -2\sin\dfrac{\pi}{3} = -\sqrt{3}$

So the gradient of the tangent to the curve at $x = \frac{\pi}{6}$ is $-\sqrt{3}$, and the gradient of the normal is $\frac{1}{\sqrt{3}}$.
Using $y - y_1 = m(x - x_1)$ to find the equations of the tangent and the normal at the point $(\frac{\pi}{6}, \frac{1}{2})$:

The equation of the tangent is $y - \dfrac{1}{2} = -\sqrt{3}\left(x - \dfrac{\pi}{6}\right)$

$$\Rightarrow \quad y = -\sqrt{3}x + \frac{1}{2} + \frac{\pi\sqrt{3}}{6}$$

The equation of the normal is $y - \dfrac{1}{2} = \dfrac{1}{\sqrt{3}}\left(x - \dfrac{\pi}{6}\right)$

$$\Rightarrow \quad y = \frac{x}{\sqrt{3}} + \frac{1}{2} - \frac{\pi}{6\sqrt{3}}$$

Check that your calculator is in 'rac mode, with a 'squa view'.

Use a graphical calculator to verify that they are the tangent and normal to the curve at $(\frac{\pi}{6}, \frac{1}{2})$.

Example 4

Find the coordinates of the stationary points on the graph of $y = \sin x + \cos x$ in the interval $0 \le x \le 2\pi$. Determine their nature. Hence sketch the graph of $y = \sin x + \cos x$.

Solution

If $y = \sin x + \cos x$, then $\frac{dy}{dx} = \cos x - \sin x$.
At stationary points on the graph, $\frac{dy}{dx} = 0$.

$$\text{So } \cos x - \sin x = 0$$
$$\Rightarrow \quad \cos x = \sin x$$
$$\Rightarrow \quad \tan x = 1$$
$$x = \tan^{-1}(1)$$

Remember that this equation has more than one solution in the interval $0 \le x \le 2\pi$. The solutions are $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.
When $x = \frac{\pi}{4}$,

$$y = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$
$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$

When $x = \frac{5\pi}{4}$,

$$y = \sin\left(\frac{5\pi}{4}\right) + \cos\left(\frac{5\pi}{4}\right)$$
$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$
$$= -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

To determine the nature of the stationary points at $(\frac{\pi}{4}, \sqrt{2})$ and $(\frac{5\pi}{4}, -\sqrt{2})$ consider the sign of the gradient of the curve at points on each side of them.
Considering the point $(\frac{\pi}{4}, \sqrt{2})$:

When $x = \frac{\pi}{6}$, $\frac{dy}{dx} = \cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}$

That is, $\frac{dy}{dx} > 0$. ◀ The function is increasing.

When $x = \frac{\pi}{3}$, $\frac{dy}{dx} = \cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}$

That is, $\frac{dy}{dx} < 0$ ◀ The function is decreasing.

This means $(\frac{\pi}{4}, \sqrt{2})$ is a maximum point on the curve.

Considering the point $(\frac{5\pi}{4}, -\sqrt{2})$:

When $x = \pi$, $\frac{dy}{dx} = \cos\pi - \sin\pi = -1 - 0$

That is, $\frac{dy}{dx} < 0$. ◀ **The function is decreasing.**

When $x = \frac{3\pi}{2}$, $\frac{dy}{dx} = \cos\left(\frac{3\pi}{2}\right) - \sin\left(\frac{3\pi}{2}\right) = 0 - (-1)$

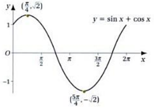
That is, $\frac{dy}{dx} > 0$. ◀ **The function is increasing.**

This means $(\frac{5\pi}{4}, -\sqrt{2})$ is a minimum point on the curve.

The graph crosses the x-axis when $y = \sin x + \cos x = 0$

$\Rightarrow \tan x = -1$

$\Rightarrow x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$ (in the stated range).



Check this result using a graphical calculator.

Graphi
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Example 5

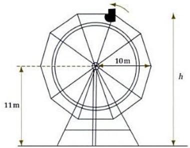
A Ferris wheel at an amusement park, with a radius of 10 m, is centred 11 m above the ground. It completes a revolution every 20 s. The height of a particular chair on the wheel varies according to

$$h = 11 + 8\sin\left(\frac{\pi t}{10}\right) + 6\cos\left(\frac{\pi t}{10}\right).$$

Find, to the nearest second, the times during the first revolution that this chair is rising, and the times during which it is falling.

Solution

When the chair is rising, height h is an increasing function of time t. This means that we need to find the times when $\frac{dh}{dt} > 0$. When the chair is falling, h is a decreasing function of time t. This means that we need to find the times when $\frac{dh}{dt} < 0$.



In order to decide when the chair is rising and falling, first locate and determine the nature of the stationary points on the height–time curve. Using the chain rule to differentiate the sine and cosine terms,

$$\begin{aligned}\frac{dh}{dt} &= 8\cos\left(\frac{\pi t}{10}\right) \times \frac{\pi}{10} - 6\sin\left(\frac{\pi t}{10}\right) \times \frac{\pi}{10} \\ &= \frac{4\pi}{5}\cos\left(\frac{\pi t}{10}\right) - \frac{3\pi}{5}\sin\left(\frac{\pi t}{10}\right)\end{aligned}$$

At stationary points on the height–time curve

$$\begin{aligned}\frac{dh}{dt} = 0 &\Rightarrow \frac{4\pi}{5}\cos\left(\frac{\pi t}{10}\right) - \frac{3\pi}{5}\sin\left(\frac{\pi t}{10}\right) = 0 \\ &\Rightarrow \frac{4\pi}{5}\cos\left(\frac{\pi t}{10}\right) = \frac{3\pi}{5}\sin\left(\frac{\pi t}{10}\right) \\ &\Rightarrow \frac{\sin\left(\frac{\pi t}{10}\right)}{\cos\left(\frac{\pi t}{10}\right)} = \frac{\left(\frac{4\pi}{5}\right)}{\left(\frac{3\pi}{5}\right)} \\ &\Rightarrow \tan\left(\frac{\pi t}{10}\right) = \frac{4}{3} \\ &\Rightarrow \frac{\pi t}{10} = \tan^{-1}\left(\frac{4}{3}\right)\end{aligned}$$

So $\frac{\pi t}{10} = 0.927, 4.069, 7.210, \ldots$
and $t = 2.95, 12.95, 22.95, \ldots$ (2 d.p.)
So during the first 20-second revolution, there are stationary points after 2.95 and 12.95 seconds. To find the nature of these stationary points, look at values of the gradient function on each side.

Considering the stationary point at $t = 2.95$ s:

When $t = 2$, $\frac{dh}{dt} = \frac{4\pi}{5}\cos\left(\frac{\pi}{5}\right) - \frac{3\pi}{5}\sin\left(\frac{\pi}{5}\right)$

That is, $\frac{dh}{dt} > 0$. ◀ **The function is increasing.**

When $t = 4$, $\frac{dh}{dt} = \frac{4\pi}{5}\cos\left(\frac{2\pi}{5}\right) - \frac{3\pi}{5}\sin\left(\frac{2\pi}{5}\right)$

That is, $\frac{dh}{dt} < 0$. ◀ **The function is decreasing.**

So $t = 2.95$ s is the location of a maximum point on the height–time curve.

Considering $t = 12.95$ s:

When $t = 12$, $\frac{dh}{dt} = \frac{4\pi}{5}\cos\left(\frac{6\pi}{5}\right) - \frac{3\pi}{5}\sin\left(\frac{6\pi}{5}\right)$

That is, $\frac{dh}{dt} < 0$. ◀ **The function is decreasing.**

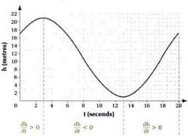
When $t = 14$, $\frac{dh}{dt} = \frac{4\pi}{5}\cos\left(\frac{7\pi}{5}\right) - \frac{3\pi}{5}\sin\left(\frac{7\pi}{5}\right)$

That is, $\frac{dh}{dt} > 0$. ◀ **The function is increasing.**

Hence, there is a minimum point when $t = 12.95$ s. Given that the wheel is centred 11 metres above the ground, and has a radius of 10 metres, the maximum and minimum heights of the chair (at these stationary points) are 21 m and 1 m respectively.

When $t = 0$, $h = 11 + 8\sin(0) + 6\cos(0) = 17$ m. Since the wheel completes a revolution every 20 seconds, the height of the chair will be 17 metres when $t = 20$.

Use this information to sketch the height-time graph. From the graph notice that, to the nearest second, the chair is rising between 0 and 3 seconds, and between 13 and 20 seconds during its first revolution. This means it is falling between $t = 3$ and $t = 13$ seconds.



5.5 Differentiation of Trigonometric Functions

Exercise

Technique

1 Differentiate each of the following with respect to x:

1 a

- **a** $y = \cos^2 x$
- **b** $y = \sin 4x + \sin 2x$
- **c** $y = \cos \frac{3}{4}x$
- **d** $y = 5\sin 2x$
- **e** $y = \cos(5 - x)$
- **f** $y = \sin^3 x$
- **g** $y = \cos^2(3x)$
- **h** $y = \sin(x^2 + 5)$
- **i** $y = 4\cos^2 \frac{x}{2}$
- **j** $y = a\sin(px) + b\cos(qx)$, where a, b, p and q are constants

2 Find the gradient of each of the following curves at the point indicated:

2 d

- **a** $y = 3\cos x$, where $x = \frac{\pi}{2}$
- **b** $y = \sin^2 x$, where $x = \frac{\pi}{4}$
- **c** $y = 6\cos\frac{x}{2}$, where $x = 3\pi$
- **d** $y = x^2 - \cos x$, where $x = \frac{\pi}{2}$

3 Find the coordinates of the points on each of the following curves at which the gradient has the value indicated:

- **a** $y = 6\sin x$, for $0 \le x \le 2\pi$, where $\frac{dy}{dx} = 6$
- **b** $y = \cos^2 x$, for $0 \le x \le \pi$, where $\frac{dy}{dx} = 1$
- **c** $y = 1 - 2\cos\frac{x}{2}$, for $0 \le x \le 2\pi$, where $\frac{dy}{dx} = \frac{1}{2}$
- **d** $y = \sin 2x$, for $0 \le x \le \pi$, where $\frac{dy}{dx} = \sqrt{2}$

4 For each of the following trigonometric functions, find an expression for $f'(x)$. Hence locate, and determine, the nature of any stationary points on the graph of $y = f(x)$ in the interval $0 \le x \le 2\pi$.

- **a** $f(x) = \sin\frac{3x}{2}$
- **b** $f(x) = \frac{1}{2}x - \cos x$
- **c** $f(x) = 2x + \sin 2x - 5$

5 For each of the following trigonometric functions, in the interval $0 \le x \le 2\pi$:

- **i** Find the coordinates of the points where the graph of $y = f(x)$ crosses the axes.
- **ii** Find the coordinates of the stationary points on the graph, and determine their nature.
- **iii** Sketch the graph of $y = f(x)$.
  Check your results using a graphical calculator.

- **a** $f(x) = \cos x - \sin x$
- **b** $f(x) = \sin x + \sqrt{3}\cos x$
- **c** $f(x) = 3\sin x - \cos x$

6 Find the equations of the tangent and normal to the curve $y = 2\cos x - 3\sin x$ at the point where:

a $x = 0$ **b** $x = \pi$

Contextual

1 The hours of daylight h in York during a calendar year can be modelled by

$$h = 12 + 2.5\cos\left[\frac{2\pi(t-172)}{365}\right],$$

where t is the number of days after the start of the year. So, $t = 1$ on 1 January and $t = 365$ on 31 December (leap years are ignored).

a Find the number of hours daylight predicted by the model on 1 June and 26 August.

b Find an expression for $\frac{dh}{dt}$.

c On which days does this model predict that the summer solstice (day with the most daylight hours) and winter solstice (day with the least daylight hours) will happen?

d Find the rate at which h is either increasing or decreasing on 28 January and on 2 November.

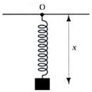
2 An object is attached to a spring, and is oscillating such that its distance x cm below some fixed point O, t seconds after its release, is given by

$$x = 20 + 8\cos(\pi t) - 5\sin(\pi t)$$

a Find an expression for $\frac{dx}{dt}$. Hence calculate the initial velocity of the object, indicating in which direction the object is moving.

b Calculate the times at which the object is momentarily at rest during its first oscillation.

c Calculate the velocity with which the object is moving the first time it moves through the midpoint of its oscillation (again indicate the direction).



Consolidation

Exercise A

1 A curve has the equation $y = 2x^3 - 3x^2 - 36x + 120$.

a Calculate the values of y when x is 3 and when x is -2.
b Find $\frac{dy}{dx}$.
c Use your expression for $\frac{dy}{dx}$ to find the coordinates of the two stationary points on the curve.
d By considering the values of $\frac{dy}{dx}$ near the stationary points, decide which type of stationary point each is.
e Sketch the curve. Deduce the range of values of h for which the equation $2x^3 - 3x^2 - 36x + 120 = h$ has three real roots.

(OCSEB)

2 A drinks machine delivers water into a cup at constant rate of $20\,\text{cm}^3\,\text{s}^{-1}$. When the height of water in the cup is h cm, the volume of water contained in the cup is $\frac{1}{30}\pi h^3\,\text{cm}^3$. Calculate the rate, in $\text{cm}\,\text{s}^{-1}$, at which the height is increasing when $h = 5$, giving your answer correct to two decimal places.

(UCLES)

3 The equation of a curve is $y = (3 - x^2)^6$. Find:

a $\frac{dy}{dx}$
b the equation of the normal at the point on the curve where $x = 2$.

(UCLES)

4 On its journey from station A to station B, a rapid transit train T passes a certain landmark C at noon. The train's journey from A to B may be modelled by the equation

$$x = \frac{1}{20}(24t - 9t^2 - 2t^3)$$

where x km denotes the displacement of T from C at t minutes past noon.

a Find the velocity of the train at t minutes past noon.
b Find the time of departure of the train from A and its time of arrival at B.
c Find the distance between the stations.
d Calculate the average speed of the train for the journey from A to B.
e Determine the greatest speed of the train during its journey.

(WJEC)

5 A curve is defined for $0 \leq x \leq 2\pi$ by the equation $y = 4\cos x + 2\cos^2 x$.

a Calculate the exact value of the gradient of the curve at the point where $x = \frac{\pi}{3}$.

b Determine the equation of the tangent to the curve at the point where $x = \frac{\pi}{3}$.

(AEB)

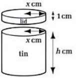
6 The two variables x and y are related by the equation $y = 3x - \frac{4}{x}$.

a Obtain an expression for $\frac{dy}{dx}$ in terms of x.

b Hence find the approximate increase in y as x increases from 2 to $2 + p$, where p is small.

(UCLES)

7 A cylindrical biscuit tin has a close-fitting lid which overlaps the tin by 1 cm, as shown. The radii of the tin and the lid are both x cm. The tin and the lid are made from a thin sheet of metal of area $80\pi\,\text{cm}^2$ and there is no wastage. The volume of the tin is $V\,\text{cm}^3$.



a Show that $V = \pi(40x - x^2 - x^3)$.

Given that x can vary:

b use differentiation to find the positive value of x for which V is stationary

c prove that this value of x gives a maximum value of V.

d Find the maximum value of V.

e Determine the percentage of the sheet metal used in the lid when V is a maximum.

(ULEAC)

8 A curve is defined by $y = x^3 - 6x^2 + 8$.

a Find an expression for $\frac{dy}{dx}$.

b Find the equation of the tangent at the point $(2, -8)$.

c Find the equation of the normal at the point where $x = 3$.

(MEI)

9 A train has to travel a distance of 60 km at a constant speed. When the train has a speed of $v\,\text{km h}^{-1}$ the running cost of the train is

$$£\left(v^2 + \frac{32\,000}{v}\right) \text{ per hour.}$$

a Find the time taken for the journey of 60 km at a constant speed of $v\,\text{km h}^{-1}$.

b Show that the total cost of the whole journey is £$(60v + \frac{1920\,000}{v^2})$.

c Find the speed at which the train should travel so that the cost of the journey is a minimum.

d Explain, briefly, why the total distance travelled does not affect the speed found in **c**.

(NICCEA)

10 An importer and distributor of computers has found an exclusive source of laptop micro-computers. They will cost her £250 per machine. In addition she will incur a cost of £5000 to adapt her distribution system to sell them, no matter how many machines she buys. The total cost of adapting her distribution system and buying n machines is £c.

a Express c in terms of n.

b Experience suggests that the number, n, of machines sold is related to the selling price per machine, s, by the equation $n = a + bs$, where a and b are constants. The importer has been informed by her market research department that if she fixes the selling price at £400 per machine she is likely to sell about 5500 machines, and if she fixes it at £500 this will fall to about 3500 machines. Find a and b based on the information supplied by the market research department.

c Show that the profit, £p, the importer will make from selling all these machines is given by $p = 18\,500s - 20s^2 - 3380\,000$.

d Find the selling price per machine which will maximise the importer's total profit and hence find the number of machines she should purchase and her total profit on selling all the machines.

(NEAB)

Exercise B

1 Find the equation of the tangent to the curve $y = (4x + 3)^5$ at the point $(-\frac{1}{2}, 1)$, giving your answer in the form $y = mx + c$.

(UCLES)

2 A storm has damaged an oil rig and caused a circular oil slick with a uniform thickness of 2 inches. The oil is spilling at a rate of 112 cubic feet per minute. Calculate the rate at which the radius of the oil slick is increasing when its radius is 50 feet. (The volume V of a cylinder of radius r and height h is given by $V = \pi r^2 h$.)

(WJEC)

3 A large tank in the shape of a cuboid is to be made from 54 m^2 of sheet metal. The tank has a horizontal rectangular base and no top. The height of the tank is x metres. Two of the opposite vertical faces are squares.

a Show that the volume, $V\ \text{m}^3$, of the tank is given by $V = 18x - \frac{2}{3}x^3$.

b Given that x can vary, use differentiation to find the maximum value of V.

c Justify that the value of V you have found is a maximum.

(ULEAC)

4 Prove that the equation of the tangent to the curve $y = x^3 - 3x^2 - 7$ at $P(3, -7)$ is $y = 9x - 34$. Find the coordinates of the other point on the curve at which the tangent is parallel to the tangent at P.

(MEI)

5 A particle moves in a straight line so that, t seconds after leaving a fixed point O, its displacement, s metres, is given by $s = \frac{1}{3}t^3 - 2t^2 + 3t$. Given that the particle returns to O when $t = T$, find the value of T. Using this value of T, find:

a the maximum displacement from O of the particle during the interval $0 \leq t \leq T$.

b the acceleration of the particle at time T seconds.

(UCLES)

6 A manufacturer wishes to make cylindrical containers to hold a dry powder. Each container has to hold $72\,\text{cm}^3$ of dry powder and has a base radius of x cm and a height of h cm.

a Write down an equation, in terms of x and h, for the volume of a container. State what assumptions you have made.

b Write down an expression, in terms of x and h, for the curved surface area of a container. Now write this expression in terms of x only.

The top is to be made of plastic and the sides and base are to be made of cardboard.

c If plastic costs 0.2 pence for $1\,\text{cm}^2$ and cardboard costs 0.1 pence for $1\,\text{cm}^2$, show that the total cost, in pence, of a container is

$$c = 0.3\pi x^2 + \frac{14.4}{x}$$

d Find the dimensions of a container so that the cost of the materials is a minimum.

(NICCEA)

7 **a** When the height of liquid in a tub is x metres the volume of liquid is $V\,\text{m}^3$, where $V = 0.05[(3x + 2)^3 - 8]$.

i Find an expression for $\frac{dV}{dx}$.

ii The liquid enters the tub at a constant rate of $0.081\,\text{m}^3\,\text{s}^{-1}$. Find the rate which the height of liquid is increasing when $V = 0.95$.

b Given that $y = \frac{8}{x^3}$ use differentiation to determine, in terms of p, where p is small, the approximate change:

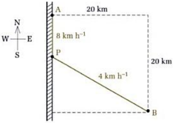
i in y as x increases from 4 to $4 + p$,

ii in x as y decreases from 1 to $1 - p$.

(UCLES)

Applications and Activities

1



As part of a training exercise, a group of soldiers have to walk from point A to point B, which lie at opposite corners of a $20\,km \times 20\,km$ square. They start off by travelling south along a track that runs along the western edge of the square. Their average speed is $8\,\text{km}\,\text{h}^{-1}$. Upon reaching some point P, they turn off the track and head directly towards B. Due to the difficult nature of the terrain they are crossing, the soldiers can only average a speed of $4\,\text{km}\,\text{h}^{-1}$ inside the square.

a Find an expression for the total time taken for this two stage journey.

b Find the position of point P, that minimises the total time taken. What is this minimum time, to the nearest minute?

Summary

- The **gradient function**, or **derivative**, of $y = f(x)$ is denoted by $\frac{dy}{dx}$ or $f'(x)$.
- **Differentiation from first principles** uses

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{f(x + \delta x) - f(x)}{\delta x} \right)$$

- 'Multiply by the power and then reduce the power by one' to differentiate powers of x.
- Differentiate the first derivative again to find the **second derivative**, the gradient of the gradient function.
- A **stationary point** is a point where $\frac{dy}{dx} = 0$.
- A **local maximum point** is a stationary point where the gradient function changes from being positive to negative.
- A **local minimum point** is a stationary point where the gradient function changes from being negative to positive.
- The gradient function on both sides of a **stationary point of inflexion** remains either positive or negative.
- Local maximum and minimum points are called **turning points**.
- If the gradient function is positive then the function is **increasing**; if it is negative the function is **decreasing**.
- The equations of the tangent and the normal to a curve $y = f(x)$ at some point (x_1, y_1) are found using $\frac{dy}{dx}$ and the result $y - y_1 = m(x - x_1)$.
- A composite function can be differentiated using the **chain rule**,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

- For a small change δy in the value of the y-coordinate corresponding to a small change δx in the x-coordinate,

$$\frac{\delta y}{\delta y} \approx \frac{dy}{dx}$$

- When the angle θ is small and measured in radians,

$$\sin\theta \approx \theta, \quad \cos\theta \approx 1 - \frac{\theta^2}{2}, \quad \text{and} \quad \tan\theta \approx \theta$$

- Trigonometric functions can be differentiated using

$$\frac{d}{dx}(\sin x) = \cos x \quad \text{and} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

Remember that x mu
be measured in radia

6 Algebra II

What you need to know

- How to use the long division algorithm (method).
- How to add and subtract proper fractions.
- How to express simple improper algebraic fractions as mixed fractions.
- The characteristics of the graphs of linear, quadratic, cubic and simple rational functions.

Review

1 Use the long division algorithm (method) to find the following, leaving any remainders as fractions:

a $4251 \div 7$ **b** $92144 \div 5$ **c** $31541 \div 12$ **d** $65235 \div 13$

2 Write each of the following in its simplest form:

a $\frac{1}{4}+\frac{1}{5}$ **b** $\frac{1}{3}+\frac{2}{9}$ **c** $\frac{3}{7}+\frac{1}{5}$ **d** $\frac{1}{4}-\frac{2}{9}$

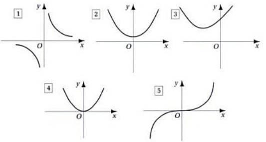
3 Express each of the following as a mixed fraction (that is, a number and a proper algebraic fraction):

Hint: First rewrite the numerator in terms of the denominator.

a $\dfrac{x+3}{x-4}$ **b** $\dfrac{x+4}{x+1}$ **c** $\dfrac{2x+3}{x+2}$ **d** $\dfrac{4x}{1-x}$

4 Match each of the following equations to the graphs:

a $y=x^2$ **b** $y=x^3$ **c** $y=x^2+1$
d $y=\frac{1}{x}$ **e** $y=(x+2)^2+1$



6.1 Polynomial Division

Without using a calculator work out $1452 \div 11$. There are many ways of doing this calculation but using the **long division algorithm** the solution takes the following form.

$$11\overline{)1452}$$

Rewrite the problem in the standard form.

$$\begin{array}{r} 1 \\ 11\overline{)1452} \end{array}$$

Find the number of 11s in 14, and subtract this result.

$$\begin{array}{r} 13 \\ 11\overline{)1452} \\ \underline{11\downarrow} \\ 35 \\ \underline{-33} \end{array}$$

Bring the next figure, 5, down to the result of the subtraction, 3. Find the number of 11s in 35 and repeat the process.

$$\begin{array}{r} 132 \\ 11\overline{)1452} \\ \underline{-11\downarrow} \\ 35 \\ \underline{-33\downarrow} \\ 22 \\ \underline{-22} \\ 0 \end{array}$$

The completed solution shows an exact answer of 132.

Use another method to check this solution.
The process for polynomial division is very similar, but instead of keeping figures in columns the powers of x are kept in columns.

Example 1

Find $\dfrac{x^3 + 4x^2 + 5x + 2}{x + 1}$.

Solution

$$x + 1\overline{)x^3 + 4x^2 + 5x + 2}$$

First rewrite the problem using the standard format for the long-division algorithm, as above.

$$\begin{array}{r} x^2\phantom{{}+4x^2+5x+2} \\ x + 1\overline{)x^3 + 4x^2 + 5x + 2} \\ -(x^3 + x^2)\phantom{{}+5x+2} \end{array}$$

Next divide x^3 by x, and subtract the result, multiplied by $(x + 1)$, from the polynomial.

x goes into x^3 exactly
times, and
$x^2(x + 1) = x^3 + x^2$.

$$\begin{array}{r} x^2 + 3x\phantom{{}+5x+2} \\ x + 1\overline{)x^3 + 4x^2 + 5x + 2} \\ \underline{-(x^3 + x^2)\ \downarrow\phantom{{}+2}} \\ 3x^2 + 5x\phantom{{}+2} \end{array}$$

Notice how the powers of x are kept under each other in straight columns. Now the subtraction takes place and the next term in the polynomial, $5x$, is brought down.

$$
\begin{array}{r}
x^2 + 3x \\
x+1 \overline{) x^3 + 4x^2 + 5x + 2} \\
\underline{-(x^3 + x^2)} \\
3x^2 + 5x \\
\underline{-(3x^2 + 3x)}
\end{array}
$$

Now repeat the process, dividing $3x^2$ by x.

x goes into $3x^2$ exactly $3x$ times, and $3x(x+1) = 3x^2 + 3x$.

$$
\begin{array}{r}
x^2 + 3x + 2 \\
x+1 \overline{) x^3 + 4x^2 + 5x + 2} \\
\underline{-(x^3 + x^2)} \\
3x^2 + 5x \\
\underline{-(3x^2 + 3x)} \\
(2x + 2) \\
\underline{-(2x + 2)} \\
0
\end{array}
$$

Repeating the process again gives the full solution.

x goes into $2x$ exactly twice, and $2(x+1) = 2x + 2$.

This means that $\dfrac{x^3 + 4x^2 + 5x + 2}{x+1} = x^2 + 3x + 2$.

Notice that this result can be checked in several ways.

1. Show that $(x+1)(x^2 + 3x + 2) = x^3 + 4x^2 + 5x + 2$.
2. Check that $\dfrac{x^3 + 4x^2 + 5x + 2}{x^2 + 3x + 2} = x + 1$.
3. Since the division is true for all values of x put $x = 10$ into each expression. What happens now? Notice how the polynomial division has produced $1452 \div 11$ again, with a result of 132. Try other values of x and see what happens.

You can also use your calculator to graph $y = \frac{x^3+4x^2+5x+2}{x+1}$ and $y = x^2 + 3x + 2$.

Example 2

Divide $x^3 + 4x - 2$ by $x - 1$.

Solution

Notice that the cubic polynomial has no x^2 term. This means the coefficient of x^2 in this polynomial must be zero.
$x^3 + 4x - 2$ can be re-written as $x^3 + 0x^2 + 4x - 2$
In this way the distinct columns for the powers of x are retained.

$$
x-1 \overline{) x^3 + 0x^2 + 4x - 2}
$$

Rewrite the problem using the standard format.

$$
\begin{array}{r}
x^2 \\
x-1 \overline{) x^3 + 0x^2 + 4x - 2} \\
\underline{-(x^3 - x^2)}
\end{array}
$$

x into x^3 goes x^2 times and $x^2(x-1) = x^3 - x^2$

$$
\begin{array}{r}
x^2 + x \\
x-1 \overline{) x^3 + 0x^2 + 4x - 2} \\
\underline{-(x^3 - x^2)} \\
x^2 + 4x \\
\underline{-(x^2 - x)}
\end{array}
$$

Subtract. Bring down the $4x$ term. x into x^2 goes x times. $x(x-1) = x^2 - x$

$0x^2 - (-x^2) = +x^2$

$$
\begin{array}{r}
x^2 + \ x + 5 \\
x - 1\overline{)x^3 + 0x^2 + 4x - 2} \\
\underline{-(x^3 - \ x^2)} \quad \downarrow \quad \\
x^2 + 4x \quad \\
\underline{-(x^2 - \ x)} \quad \\
5x - 2 \\
\underline{-(5x - 5)} \\
3
\end{array}
$$

Repeat the process. Notice that the final subtraction does not give a zero result.

Since the result of the final subtraction is 3 the original division problem must have a remainder of 3.

$$\frac{x^3 + 4x - 2}{x - 1} = (x^2 + x + 5) \text{ with remainder } 3$$

which can be rewritten

$$\frac{x^3 + 4x - 2}{x - 1} = (x^2 + x + 5) + \frac{3}{x - 1}$$

Notice how the remainder term has the same divisor as the original problem.

Check this result by:

1. Multiplying $(x^2 + x + 5)$ by $(x - 1)$ and then adding the remainder of 3. The result should be $x^3 + 4x - 2$
2. Substituting $x = 10$ and showing that $1398 \div 9$ has a remainder of 3.

Example 3 demonstrates that polynomial division can create an answer with two distinct components:

- a polynomial, whose degree, or order, is less that the degree of the polynomial in the problem – this term is called the **quotient**
- an algebraic fraction, whose denominator is the divisor of the original problem and whose numerator is called the **remainder**.

So $\dfrac{x^3 + 4x - 2}{x - 1}$ has a quotient of $(x^2 + x + 5)$ and a remainder of 3.

Example 3

Find the quotient and remainder for $(4x^3 + 2x^2 - 8x + 6)$ divided by $(2x - 3)$.

Solution

These can be found by using the long division algorithm (method) for $\dfrac{4x^3 + 2x^2 - 8x + 6}{2x - 3}$.

$$\begin{array}{r}
2x^2 + 4x + 2 \\
2x-3 \overline{\smash{)}\,4x^3 + 2x^2 - 8x + 6} \\
\underline{4x^3 - 6x^2} \phantom{- 8x + 6} \\
8x^2 - 8x \\
\underline{8x^2 - 12x} \\
4x + 6 \\
\underline{4x - 6} \\
12
\end{array}$$

Since

$$\frac{4x^3 + 2x^2 - 8x + 6}{2x - 3} = (2x^2 + 4x + 2) + \frac{12}{(2x - 3)}$$

the quotient is $(2x^2 + 4x + 2)$ and the remainder is 12.

Check this result by substituting $x = 10$ to show $4126 \div 17$ has a remainder of 12.

The remainder theorem

When a polynomial $P(x)$ is divided by a linear factor $(x - a)$ then the remainder is $P(a)$.

Learn this important result and follow the argument used in its proof.

This is a useful result if information about the remainder is needed from polynomial division.

Proof of the remainder theorem

Let $P(x)$ be a polynomial of degree n where $n \geq 2$. Then
$P(x) \equiv (x - a)Q(x) + R$, where $Q(x)$ is the quotient and R is the remainder.
This identity is true for all x, and so it is true for $x = a$.
When $x = a$, this becomes $P(a) = (a - a)Q(a) + R$.
Since $(a - a) = 0$, the remainder $R = P(a)$.

Recall that '$\equiv$' represents an identity.

Example 4

Use the remainder theorem to find the remainders when:

a $3x^2 + 2x - 5$ is divided by $x - 4$

b $x^3 - 2x^2 - 3x + 1$ is divided by $2x + 1$.

Solution

a Let $P(x) = 3x^2 + 2x - 5$.
The remainder when $P(x)$ is divided by $(x - 4)$ is $P(4)$.

$$\begin{aligned} R = P(4) &= 3(4)^2 + 2(4) - 5 \\ &= 48 + 8 - 5 = 51 \end{aligned}$$

b Let $P(x) = x^3 - 2x^2 - 3x + 1$.
The remainder when $P(x)$ is divided by $(2x + 1)$ is $P(-\frac{1}{2})$.

$$\begin{aligned} R = P(-\tfrac{1}{2}) &= (-\tfrac{1}{2})^3 - 2(-\tfrac{1}{2})^2 - 3(-\tfrac{1}{2}) + 1 \\ &= -\tfrac{1}{8} - \tfrac{2}{4} + \tfrac{3}{2} + 1 = \tfrac{15}{8} \end{aligned}$$

Check these results using the long division algorithm (method).

Example 5

When $5x^2 + x - 8k$ is divided by $(x - 1)$ the remainder is 2. Find k.

Solution

Let $P(x) = 5x^2 + x - 8k$.

$$P(1) = 5 + 1 - 8k = 6 - 8k$$

But $P(1) = R$, where R is the remainder when $P(x)$ is divided by $(x - 1)$. Since $R = 2$, it follows that $6 - 8k = 2$

By the remainder theorem.

$$\Rightarrow \quad 8k = 4$$

$$\Rightarrow \quad k = \tfrac{1}{2}$$

The factor theorem

The remainder theorem can also be used to provide a useful result for analysing higher order polynomials. If the remainder from dividing the polynomial $P(x)$ by $(x - a)$ is zero, then the linear term $(x - a)$ must be a factor of the polynomial $P(x)$.

If $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.

Example 6

Factorise $x^3 + 2x^2 - 5x - 6$.

Solution

Let $P(x) = x^3 + 2x^2 - 5x - 6$, a cubic polynomial of order 3. To find the linear factors of $P(x)$, if any exist, substitute values of x until $P(x) = 0$. Adopt a trial and error approach, substituting factors of the constant term, -6, in $P(x)$.

$P(x)$ could have no factors, one linear an one quadratic or thre linear factors.

$$P(1) = 1^3 + 2(1)^2 - 5(1) - 6 \neq 0$$
$$P(2) = 2^3 + 2(2)^2 - 5(2) - 6 = 8 + 8 - 10 - 6 = 0$$

Since $P(2) = 0$, $(x - 2)$ is a factor of $P(x)$.

$$P(3) = 3^3 + 2(3)^2 - 5(3) - 6 \neq 0$$
$$P(6) = 6^3 + 2(6)^2 - 5(6) - 6 \neq 0$$
$$P(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6 = -1 + 2 + 5 - 6 = 0$$

Since $P(-1) = 0$, $(x + 1)$ is a factor of $P(x)$.

$$P(-2) = (-2)^3 + 2(-2)^2 - 5(-2) - 6 \neq 0$$
$$P(-3) = (-3)^3 + 2(-3)^2 - 5(-3) - 6 = -27 + 18 + 15 - 6 = 0$$

Since $P(-3) = 0$, $(x + 3)$ is also a factor of $P(x)$.

So $P(x) = (x - 2)(x + 1)(x + 3)$ ◀ **Check that $(x - 2)(x + 1)(x + 3) = x^3 + 2x^2 - 5x - 6$.**

It may not be necessary to keep substituting values of x until all the factors have been found. Once one factor has been found the polynomial $P(x)$ can be rewritten as $P(x) = (x - a)Q(x)$, and the resulting quotient $Q(x)$ may then be easier to factorise.

Example 7

Factorise $3x^3 + 2x^2 - 19x + 6$.

Solution

Let $P(x) = 3x^3 + 2x^2 - 19x + 6$.

$$P(1) = 3(1)^3 + 2(1)^2 - 19(1) + 6 \neq 0$$

$$P(2) = 3(2)^3 + 2(2)^2 - 19(2) + 6 = 24 + 8 - 38 + 6 = 0$$

Try substituting factors of the constant term, 6, into P(x) until P(x)= 0 and a linear factor can be identified.

Since $P(2) = 0$, $(x - 2)$ is a factor of $P(x)$.

So $P(x) = (x - 2)Q(x)$

The quotient $Q(x)$ could be found by using the long division algorithm or by using a technique known as **algebraic juggling** and then equating coefficients. The juggling method uses both terms of the linear factor to find the quotient by comparing coefficients of the polynomials on each side of the equals sign as follows.

$$\begin{aligned} P(x) &= (x - 2)Q(x) \\ \Rightarrow 3x^3 + 2x^2 - 19x + 6 &= (x - 2)(ax^2 + bx + c) \\ &= ax^3 + bx^2 + cx - 2ax^2 - 2bx - 2c \\ &= ax^3 + (b - 2a)x^2 + (c - 2b)x - 2c \end{aligned}$$

Comparing coefficients of x^3, $a = 3$.
Comparing coefficients of x^2, $b - (2 \times 3) = b - 6 = 2$. So $b = 8$.
Comparing constant terms, $-2c = 6$. So $c = -3$.

So $3x^3 + 2x^2 - 19x + 6 = (x - 2)(3x^2 + 8x - 3)$

Check this by expanding the RHS.

The quotient can now be factorised by using either the factor theorem again or by another appropriate method. Since $Q(x)$ is a quadratic it can be factorised using PAFF.

$$Q(x) = 3x^2 + 8x - 3$$

P: $3 \times (-3) = -9$ A: $+8$ F: $9, -1$

$$\begin{aligned} Q(x) = 3x^2 + 8x - 3 &= 3x^2 + 9x - x - 3 \\ &= 3x(x + 3) - 1(x + 3) \\ &= (x + 3)(3x - 1) \end{aligned}$$

P is the constant term multiplied by the coefficient of x^2. A is the number of 'x's and F values add to A and multiply to P.

Having factorised $Q(x)$ the original polynomial $P(x)$ can be written in a factorised form as

$$\begin{aligned} P(x) &= 3x^3 + 2x^2 - 19x + 6 \\ &= (x-2)Q(x) \\ &= (x-2)(x+3)(3x-1) \end{aligned}$$

Writing cubic polynomials in their factorised form is particularly useful if the graph needs to be sketched or related inequalities solved.

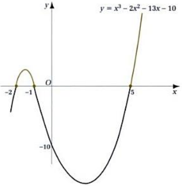
Example 8

Sketch the graph of $y = x^3 - 2x^2 - 13x - 10$. Hence, or otherwise, state the range of values of x for which $y \geq 0$.

Solution

Use the factor theorem first to show that,
$y = x^3 - 2x^2 - 13x - 10 = (x+2)(x+1)(x-5)$.
So $y = 0$ when $x = -2, -1, 5$; the graph cuts the x-axis at $(-2, 0)$, $(-1, 0)$ and $(5, 0)$.
The graph cuts the y-axis when $x = 0$ at the point $(0, -10)$.
Given the general shape of the cubic graph and the four points through which it must pass a sketch can be made. ◀ **Check this result using a graphical calculator.**

The values of x for which $y \geq 0$ can be read from the graph; $y \geq 0$ when $-2 \leq x \leq -1$ and $x \geq 5$.



Remember that this sketch, and the axes not scaled accuratel

The remainder and factor theorems can also be used to find the value of unknown coefficients in a polynomial. This often creates simultaneous equations.

Example 9

The polynomial $2x^3 + ax^2 + bx + 8$ has $(x - 1)$ as a factor and gives a remainder of 50 when divided by $(x - 3)$. Find a and b.

Solution

Let $P(x) = 2x^3 + ax^2 + bx + 8$
Since $(x - 1)$ is a factor, $P(1) = 0$.

$$P(1) = 2 + a + b + 8 = 0$$
$$a + b = -10 \qquad [1]$$

When $P(x)$ is divided by $(x - 3)$ there is a remainder of 50. This means $P(3) = 50$.

$$P(3) = 2.3^3 + a.3^2 + b.3 + 8 = 50$$
$$54 + 9a + 3b + 8 = 50$$
$$9a + 3b = -12 \qquad [2]$$

Equations [1] and [2] are simultaneous equations. Using the elimination technique we can find a.

From equation [2],	$9a + 3b = -12$
3 × equation [1]	$3a + 3b = -30$
Subtracting,	$6a = 18$
$\Rightarrow$	$a = 3$
Put $a = 3$ in equation [1]	$3 + b = -10$
$\Rightarrow$	$b = -13$

So $P(x) = 2x^3 + 3x^2 - 13x + 8$, with $a = 3$ and $b = -13$.

6.1 Polynomial Division
Exercise
Technique

1 Use the long division algorithm to find:

a $\dfrac{x^3 + 3x^2 + 5x + 3}{x + 1}$ b $\dfrac{x^3 + x^2 - x - 10}{x - 2}$

c $\dfrac{x^3 - x^2 - 9x - 6}{x + 2}$ d $\dfrac{2x^3 - x^2 + 4x + 15}{2x + 3}$

2 Use the remainder theorem to find the remainder in the following:

a $\dfrac{x^2 + 2x - 5}{x - 2}$ b $\dfrac{3x^2 + 2x + 2}{x + 1}$

c $\dfrac{2x^2 + 3x - 3}{2x + 1}$ d $\dfrac{4x^2 + 2x - 1}{2x - 3}$

2 a

3 Factorise completely the following polynomials:

a $x^3 - x^2 - 4x + 4$ d $x^3 + 5x^2 - 8x - 12$

b $x^3 - 6x^2 + 11x - 6$ e $2x^3 + 13x^2 + 22x + 8$

c $x^3 - 4x^2 - x + 4$ f $2x^3 + 5x^2 + x - 2$

Hint: Use the factor theorem first.

3 b

4 Solve the following inequalities:

a $x^3 - 12x^2 + 39x - 28 > 0$ d $x^3 - x^2 - 30x \geq 0$

b $x^3 - 10x^2 + 11x + 70 < 0$ e $x^3 + x^2 - 12x < 0$

c $x^3 - 6x^2 - 13x + 42 \leq 0$ f $2x^3 - 7x^2 - 46x - 21 > 0$

4 e

Contextual

1 Let $f(x) = x^4 - 2x^3 - 12x^2 + 40x - 32$.

a Factorise $f(x)$ completely. b When is $f(x) > 0$?

2 $(x + 1)$ and $(x - 2)$ are both factors of $x^3 + ax^2 + bx - 6$. Find a, b and the third linear factor.

3 Find the coordinates of the points where $y = x^3 - 3x^2 - 16x - 12$ crosses the x-axis.

4 When $x^3 + ax^2 + bx - 1$ is divided by $(x - 2)$ and $(x + 2)$ the remainders are 27 and 3 respectively. Find a and b.

5 Solve the equation $6x^3 - 7x^2 - 9x - 2 = 0$.

Hint: First write the expression as the product of three linear factors.

6 If $f(x) = 4x^3 - 19x^2 + 19x + 6$, find the range of values of x for which $f(x) \leq 0$.

6.2 Algebraic Fractions

An algebraic fraction is a fraction that contains an algebraic term. Some examples are

$$\frac{x}{3},\ \frac{2}{y},\ \frac{3}{4x+1},\ \frac{6t+7}{t^2+3t-1}$$

Algebraic fractions can be combined in the same way as numerical fractions, and obey the same rules for addition, subtraction, multiplication and division.

Addition and subtraction

Find $\frac{1}{3}+\frac{2}{5}$. Check that you get a result of $\frac{11}{15}$. How did you arrive at this result? When adding fractions the first step is to find a common denominator. This is usually the least (or lowest) common multiple (LCM) of the denominators of the fractions being added. Having found a common denominator each fraction is then written in an equivalent form using this new denominator. The numerators can then be added to give the solution to the problem.

Here the LCM of 3 and 5 is 15 but 15, 30, 45, ... would serve equally well as denominators.

$$\tfrac{1}{3}+\tfrac{2}{5}=\tfrac{5}{15}+\tfrac{6}{15}=\tfrac{11}{15}$$

Common denominator of 15 – the LCM of 3 and 5. Notice the new equivalent fractions.

The same technique can be used to add and subtract algebraic fractions.

Example 1

a Find $\frac{2}{(x-3)}+\frac{3}{(x+5)}$. **b** Find $\frac{10}{(2x-1)}-\frac{5}{(x+3)}$.

Solution

a The common denominator will be $(x-3)(x+5)$. Notice that this is the LCM of $(x-3)$ and $(x+5)$.

$$\frac{2}{(x-3)}+\frac{3}{(x+5)}=\frac{2(x+5)}{(x-3)(x+5)}+\frac{3(x-3)}{(x-3)(x+5)}$$

Rewrite the fractions using the common denominator.

$$=\frac{2x+10+3x-9}{(x-3)(x+5)}$$

Expand the brackets and collect like terms.

$$=\frac{5x+1}{(x-3)(x+5)}$$

b The common denominator will be $(2x-1)(x+3)$.

$$\frac{10}{(2x-1)}-\frac{5}{(x+3)}=\frac{10(x+3)-5(2x-1)}{(2x-1)(x+3)}$$

$$=\frac{10x+30-10x+5}{(2x-1)(x+3)}$$

$$=\frac{35}{(2x-1)(x+3)}$$

Example 2

Express as a single fraction, $\frac{3}{(x+2)}+\frac{2x-1}{(x^2+4x+1)}$.

Solution

Again notice that both algebraic fractions are proper. Notice also that the quadratic expression in the denominator will not factorise. The same technique can be used, even though one of the denominators is quadratic and not linear.

The order (degree) of polynomial in the numerator is less tha[n] order (degree) of the polynomial in the denominator.

$$\begin{aligned}\frac{3}{(x+2)}+\frac{2x-1}{(x^2+4x+1)} &= \frac{3(x^2+4x+1)+(2x-1)(x+2)}{(x+2)(x^2+4x+1)}\\ &= \frac{(3x^2+12x+3)+(2x^2+4x-x-2)}{(x+2)(x^2+4x+1)}\\ &= \frac{5x^2+15x+1}{(x+2)(x^2+4x+1)}\end{aligned}$$

Check this result by substituting $x = 10$.

Example 3

Express $\frac{3}{x+2}-\frac{1}{(x+2)^2}$ as a single fraction.

Solution

Notice that the term $(x+2)$ appears in both denominators and that the LCM of $(x+2)$ and $(x+2)^2$ is $(x+2)^2$. This means the common denominator will also be $(x+2)^2$.

$$\begin{aligned}\frac{3}{x+2}-\frac{1}{(x+2)^2} &= \frac{3(x+2)}{(x+2)^2}-\frac{1}{(x+2)^2}\\ &= \frac{3(x+2)-1}{(x+2)^2}\\ &= \frac{3x+6-1}{(x+2)^2}=\frac{3x+5}{(x+2)^2}\end{aligned}$$

Multiplication and division

Find $\frac{2}{3}\times\frac{6}{7}$. Check that you can get a result of $\frac{4}{7}$. How did you arrive at this result? When multiplying fractions the first step is to find if there is a common factor in the numerator and the denominator. If a common factor does exist, it can be 'cancelled down'. Then multiply the numerators and denominators separately to give the numerator and denominator of the result.

$$\frac{2}{3}\times\frac{6}{7}=\frac{2}{{}_1\cancel{3}}\times\frac{\cancel{6}^{\,2}}{7}=\frac{2\times 2}{1\times 7}=\frac{4}{7}$$

6 and 3 have a comm[on] factor 3, so it can be 'cancelled'.

The same technique can be used to multiply algebraic fractions.

Example 4

a Simplify $\frac{3(x+5)}{7} \times \frac{1}{6x}$.

b Express as a single fraction, $\frac{2(x+3)}{5} \times \frac{1}{(x^2-9)}$.

c Simplify $\frac{3x-12}{10} \times \frac{5x}{2x-8}$.

Solution

a $\frac{3(x+5)}{7} \times \frac{1}{6x} = \frac{{}^{1}\cancel{3}(x+5)}{7} \times \frac{1}{{}_{2}\cancel{6}x}$

$= \frac{x+5}{14x}$

The numerator and the denominator have the common factor 3.

b $\frac{2(x+3)}{5} \times \frac{1}{(x^2-9)} = \frac{2(x+3)}{5} \times \frac{1}{(x-3)(x+3)}$ ◀ **Difference of two squares.**

$= \frac{2\cancel{(x+3)}^{1}}{5} \times \frac{1}{(x-3)\cancel{(x+3)}_{1}}$

$= \frac{2}{5(x-3)}$

Factorise the quadratic expression
The common factor is $(x+3)$.

c $\frac{3x-12}{10} \times \frac{5x}{2x-8} = \frac{3(x-4)}{10} \times \frac{5x}{2(x-4)}$

$= \frac{3\cancel{(x-4)}^{1}}{{}_{2}\cancel{10}} \times \frac{{}^{1}\cancel{5}x}{2\cancel{(x-4)}_{1}}$

$= \frac{3x}{4}$

Factorise the terms in the fractions first.

There are two common factors; $(x-4)$ and 5.

The division of fractions can be tackled by reducing the problem to multiplication. Remember that to divide fractions, invert the second fraction and change the sign from division to multiplication. For example, $\frac{2}{3} \div \frac{7}{6} = \frac{2}{3} \times \frac{6}{7} = \frac{4}{7}$. The same technique can be applied to algebraic fractions.

Check this result using a calculator with fraction keys.

Example 5

a Simplify $\frac{5x}{3} \div \frac{11x}{6}$.

b Express as a single fraction $\frac{3x-6}{8x} \div \frac{5x^2-5x-10}{6x}$.

Solution

a $$\frac{5x}{3} \div \frac{11x}{6} = \frac{5x}{3} \times \frac{6}{11x}$$

$$= \frac{5\cancel{x}^{1}}{{}_{1}\cancel{3}} \times \frac{\cancel{6}^{2}}{11\cancel{x}_{1}}$$

$$= \frac{10}{11}$$

b $$\frac{3x-6}{8x} \div \frac{5x^2-5x-10}{6x} = \frac{3(x-2)}{8x} \div \frac{5(x-2)(x+1)}{6x}$$ ◀ **Factorising.**

$$= \frac{3(x-2)}{8x} \times \frac{6x}{5(x-2)(x+1)}$$

Inverting the second fraction and multiplying.

$$= \frac{3\cancel{(x-2)}}{{}_{4}\cancel{8x}} \times \frac{{}^{3}\cancel{6x}}{5\cancel{(x-2)}(x+1)}$$

$$= \frac{9}{20(x+1)}$$

6.2 Algebraic Fractions

Exercise

Technique

1 Find:

a $\dfrac{3}{x-1}+\dfrac{5}{x+2}$

b $\dfrac{4}{x-2}+\dfrac{6}{x+2}$

c $\dfrac{6}{x-2}-\dfrac{6}{x+3}$

d $\dfrac{3}{x-1}-\dfrac{2}{x+1}$

e $\dfrac{7}{x+1}-\dfrac{3}{x-1}$

f $\dfrac{1}{x+1}+\dfrac{4}{x+2}$

2 Express each of the following as single fractions:

a $\dfrac{2}{3-2x}-\dfrac{3}{2x+1}$

b $\dfrac{2}{x-1}+\dfrac{5}{4x+3}$

c $\dfrac{4}{2x+1}-\dfrac{2}{x-2}$

d $\dfrac{3}{x-2}+\dfrac{2}{7x+1}$

e $\dfrac{2}{3x-2}+\dfrac{1}{x-1}$

f $\dfrac{3}{3x-2}-\dfrac{2}{1-2x}$

$\boxed{2}$ c

3 Identify the LCM of these denominators and use it to write each of the following as a single algebraic fraction:

a $\dfrac{2x+1}{x^2+x-12}+\dfrac{2x+3}{(x+4)(x-2)}$

b $\dfrac{x}{2x^2+3x-2}-\dfrac{3}{x+1}$

c $\dfrac{1}{x+1}+\dfrac{3}{x^2+2x+1}$

d $\dfrac{3}{x^2-9}-\dfrac{2}{x-3}$

e $\dfrac{3}{x^2+x-2}-\dfrac{2}{3x-1}$

f $\dfrac{2}{3(x+2)}+\dfrac{2}{(x+2)^2}$

$\boxed{3}$ d

4 Express each of the following as a single fraction:

a $\dfrac{3x}{8}\times\dfrac{2}{x}$

b $\dfrac{5(x-1)}{3}\times\dfrac{2}{15x}$

c $\dfrac{3(x-7)}{6}\times\dfrac{1}{(x^2-49)}$

d $\dfrac{2x-6}{7}\times\dfrac{2x}{4(x^2-9)}$

e $\dfrac{(x+1)}{14}\times\dfrac{2}{(x^2-2x-3)}$

f $\dfrac{3}{x+2}\times\dfrac{x^2+4x+4}{6x}$

$\boxed{4}$ f

5 Simplify:

a $\dfrac{2}{5x}\div\dfrac{8}{x}$

b $\dfrac{2x}{9}\div\dfrac{8(x+1)}{3}$

c $\dfrac{x-1}{15}\div\dfrac{x^2-1}{3}$

d $\dfrac{3x-6}{2}\div\dfrac{6(x^2-4)}{x}$

e $\dfrac{x+3}{9}\div\dfrac{x^2+2x-3}{3}$

f $\dfrac{2x}{x-7}\div\dfrac{4}{x^2-8x+7}$

6.3 Partial Fractions

Two or more proper fractions can be combined to give a single fraction. For example $\frac{1}{3}+\frac{2}{5}=\frac{11}{15}$. Equally, a fraction can be expressed as a sum or difference of two or more proper fractions, known as **partial fractions**.

We can usually apply this technique to algebraic fractions, and it is sometimes quite advantageous to write an algebraic fraction as a sum or difference of simpler algebraic fractions (see Chapter 12). The key to this process lies in the factorisation of the denominator. We will consider three categories of denominators:

- linear factors in the denominator
- quadratic factors in the denominator
- repeated factors in the denominator.

Linear factors in the denominator

Look at the fraction $\dfrac{5x+1}{(x-3)(x+5)}$.

The denominator has been factorised into two linear factors, $(x-3)$ and $(x+5)$. This suggests that this algebraic fraction could be rewritten as

$$\frac{5x+1}{(x-3)(x+5)}\equiv\frac{A}{(x-3)}+\frac{B}{(x+5)},$$

where A and B are constants. Checking back to Section 6.2, Example **1a**, you will find that $A=2$ and $B=3$ provide the solution.

$$\frac{5x+1}{(x-3)(x+5)}\equiv\frac{2}{(x-3)}+\frac{3}{(x+5)}$$

Two techniques can be used to find the constants that form the numerators of the partial fractions. The first uses the following steps:

Step ① Identify the linear factors in the denominator.
Step ② Write each linear factor as a new denominator with a constant term numerator.
Step ③ Add the algebraic fractions together and then equate the two numerators.
Step ④ Substitute values of x that make the coefficients of A and B zero, in turn, and solve the resulting equation.

Example 1

Write $\dfrac{5x+7}{(x+3)(x-1)}$ in partial fractions.

Solution

Notice that the denominator has been factorised into two linear factors, $(x+3)$ and $(x-1)$. This suggests that

$$\frac{5x+7}{(x+3)(x-1)} \equiv \frac{A}{(x+3)} + \frac{B}{(x-1)}$$ ◄ ② **Write each linear factor as a new fraction.**

Now add together the algebraic fractions on the RHS of the identity.

Recall that RHS means right-hand side.

$$\frac{5x+7}{(x+3)(x-1)} \equiv \frac{A(x-1)+B(x+3)}{(x+3)(x-1)}$$ ◄ ③ **Add the fractions.**

Since the denominators on both sides are equal it follows that the two numerators must be equivalent to each other.

So $\quad 5x+7 \equiv A(x-1)+B(x+3)$ ◄ ③ **Equate the numerators.**

This statement is true for all values of x, so A can be found by substituting values of x that reduce the coefficient of B to zero, and vice versa.

When $x = 1$, $\quad 5+7 = A(0)+B(4)$ ◄ ④ **Substitute values of x.**

$$\Rightarrow \quad 12 = 4B$$
$$\Rightarrow \quad B = 3$$

When $x = -3$, $\quad -15+7 = A(-3-1)+B(0)$

$$\Rightarrow \quad -8 = -4A$$
$$\Rightarrow \quad A = 2$$

So $\dfrac{5x+7}{(x+3)(x-1)} \equiv \dfrac{2}{(x+3)} + \dfrac{3}{(x-1)}$

Check this result by adding these two algebraic partial fractions together.

Example 2

Express $\dfrac{3}{x^2-x-2}$ in partial fractions.

Solution

The denominator of this fraction is a quadratic. Since it has not been written as a product of linear factors the first step is to factorise the denominator.

Check that $x^2-x-2 = (x+1)(x-2)$. ◄ ① **Identify the linear factors.**

So $\dfrac{3}{x^2-x-2} \equiv \dfrac{3}{(x+1)(x-2)}$

Now the denominator contains two distinct linear factors.

$$\frac{3}{(x+1)(x-2)} \equiv \frac{A}{(x+1)} + \frac{B}{(x-2)}$$ ◄ ② **Use each linear factor to write down a new fraction.**

Adding together the algebraic partial fractions,

$$\frac{3}{(x+1)(x-2)} \equiv \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$ ◀ ③ **Add the fractions.**

Equating the numerators,

$$3 \equiv A(x-2)+B(x+1)$$ ◀ ③ **Equate the numerators.**

When $x = 2$, $\quad 3 = A(0) + B(3)$ ◀ ④ **Substitute values of x.**

$$\Rightarrow 3 = 3B$$
$$\Rightarrow B = 1$$

When $x = -1$, $\quad 3 = A(-1-2) + B(0)$

$$\Rightarrow 3 = -3A$$
$$\Rightarrow A = -1$$

So $\dfrac{3}{x^2-x-2} \equiv -\dfrac{1}{(x+1)} + \dfrac{1}{(x-2)}$

Check this result by combining the terms the RHS of the equivalence.

Step ④ provides the second method of finding partial fractions. It is known as the **'cover-up' rule**. The cover-up rule uses the fact that some values of x create a zero in the denominator. From above, we have seen that these values are also the ones that allow us to calculate the required numerators in the partial fractions. By covering-up these factors the whole process can be speeded up, and can often be calculated mentally.

Example 3

Use the cover-up rule to express $\dfrac{5x+7}{(x+3)(x-1)}$ in partial fractions.

Solution

Look at the fraction $\dfrac{5x+7}{(x+3)(x-1)}$.

Notice that this is another way of tackl Example 1.

Notice that $(x+3) = 0$ when $x = -3$.
Cover up $(x+3)$ and substitute $x = -3$ in the parts of the fraction you can still see.

$$\frac{5x+7}{(\quad\;)(x-1)} = \frac{-15+7}{-4} = \frac{-8}{-4} = 2$$

The solution 2 is fou when $(x+3)$ is covered up.

Also notice that $(x-1) = 0$ when $x = 1$.
Cover up $(x-1)$ and substitute $x = 1$ in the parts of the fraction you can still see.

$$\frac{5x+7}{(x+3)(\quad\;)} = \frac{5+7}{4} = \frac{12}{4} = 3$$

The solution 3 is fou when $(x-1)$ is covered up.

So $\dfrac{5x+7}{(x+3)(x-1)} \equiv \dfrac{2}{(x+3)} + \dfrac{3}{(x-1)}$

Notice how the number found when 'covering-up' becomes the numerator above that 'covered' linear factor.

This means the cover-up rule can only be used with linear denominators.

Example 4

Express $\dfrac{5x^2+3x-14}{(x-2)(x-1)(x+4)}$ in partial fractions.

Solution

Notice that the denominator has three linear factors. Using the cover-up rule this can be put into partial fractions in three steps.
Covering up $(x-2)$ and substituting $x = 2$,

$$\frac{5x^2+3x-14}{\boxed{(\quad)}(x-1)(x+1)} = \frac{20+6-14}{1\times 3} = \frac{12}{3} = 4$$

Covering up $(x-1)$ and substituting $x = 1$,

$$\frac{5x^2+3x-14}{(x-2)\boxed{(\quad)}(x+1)} = \frac{5+3-14}{-1\times 2} = \frac{-6}{-2} = 3$$

Covering up $(x+1)$ and substituting $x = -1$,

$$\frac{5x^2+3x-14}{(x-2)(x-1)\boxed{(\quad)}} = \frac{5-3-14}{-3\times -2} = \frac{-12}{6} = -2$$

So $\dfrac{5x^2+3x-14}{(x-2)(x-1)(x+4)} \equiv \dfrac{4}{(x-2)} + \dfrac{3}{(x-1)} - \dfrac{2}{(x+1)}$

Check this result by adding the algebraic fractions.

Quadratic factors in the denominators

Some algebraic fractions have denominators that contain a quadratic factor that doesn't factorise; for example, $\dfrac{3x^2+5x+6}{(x+1)(x^2+3)}$.
Notice that the denominator has a linear factor $(x+1)$, and a quadratic factor (x^2+3), which doesn't factorise.
What happens when fractions of this type are written in partial fractions?
The linear factor has a constant numerator, A, and the quadratic factor has a linear numerator of the form $(Bx+C)$.

Example 5

Express $\dfrac{3x^2+5x+6}{(x+1)(x^2+3)}$ in partial fractions.

Solution

This fraction can be written as the sum of two distinct fractions; one with $(x+1)$ as its denominator and the other with (x^2+3) as its denominator. The latter fraction has a linear numerator.

$$\frac{3x^2+5x+6}{(x+1)(x^2+3)} \equiv \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+3)}$$

Now add together the algebraic fractions on the RHS.

$$\frac{3x^2+5x+6}{(x+1)(x^2+3)} \equiv \frac{A(x^2+3)+(Bx+C)(x+1)}{(x+1)(x^2+3)}$$

Since the denominators on both sides are equal the two numerators must be equivalent.

$$3x^2+5x+6 \equiv A(x^2+3)+(Bx+C)(x+1)$$

A, B and C can now be found from a combination of:

- substituting suitable values of x
- equating coefficients of powers of x.

Let $x = -1$

Then $x+1=0$, and \
eliminate the term \
$(Bx+C)$.

$$\begin{aligned} \text{Then }\ 3(-1)^2+5(-1)+6 &= A[(-1)^2+3]+(-B+C)(0) \\ \Rightarrow\qquad 3-5+6 &= 4A+0 \\ \Rightarrow\qquad 4 &= 4A \\ \Rightarrow\qquad A &= 1 \end{aligned}$$

Let $x = 0$

This eliminates the te \
Bx.

$$\begin{aligned} \text{Then }\ 3(0)^2+5(0)+6 &= A(0^2+3)+C(0+1) \\ \Rightarrow\qquad 6 &= 3A+C \end{aligned}$$

Substituting $A=1$, $6=3+C$

$$\Rightarrow\qquad C = 3$$

Returning to the equivalence between the numerators,

$$\begin{aligned} 3x^2+5x+6 &\equiv A(x^2+3)+(Bx+C)(x+1) \\ &= Ax^2+3A+Bx^2+Bx+Cx+C \\ &= (A+B)x^2+(B+C)x+(3A+C) \end{aligned}$$

Equating the coefficients of the different powers of x:

$$x^2: \quad 3 = A + B \qquad [1]$$

$$x: \quad 5 = B + C \qquad [2]$$

$$\text{constant terms:} \quad 6 = 3A + C \qquad [3]$$

We know that $A = 1$ and $C = 3$, so the value of B can be established from equation [1] and checked in equation [2], or vice versa.
From [1], $3 = 1 + B \Rightarrow B = 2$
In [2], $5 = B + C = 2 + 3 = 5$
Having found $A = 1$, $B = 2$ and $C = 3$, the original fraction can be rewritten in terms of partial fractions as:

$$\frac{3x^2 + 5x + 6}{(x+1)(x^2+3)} \equiv \frac{1}{(x+1)} + \frac{2x+3}{(x^2+3)}$$

Check this result by:
(i) adding the two algebraic fractions;
(ii) substituting $x = 10$.

Example 6

Express $\dfrac{3x^2 + 2x + 3}{(x+1)(x^2+3)}$ in partial fractions.

Solution

$$\frac{3x^2 + 2x + 3}{(x+1)(x^2+3)} \equiv \frac{A}{(x+1)} + \frac{Bx + C}{(x^2+3)}$$

$$\equiv \frac{A(x^2+3) + (Bx+C)(x+1)}{(x+1)(x^2+3)}$$

Equating the numerators,

$$3x^2 + 2x + 3 \equiv A(x^2+3) + (Bx+C)(x+1)$$

When $x = -1$, ◀ **Substitute suitable values of x.**

So $(x + 1) = 0$ and we eliminate the term $(Bx + C)$.

$$3(-1)^2 + 2(-1) + 3 = A[(-1)^2 + 3]$$

$$\Rightarrow 3 - 2 + 3 = 4A$$

$$\Rightarrow \quad 4 = 4A$$

$$\Rightarrow \quad A = 1$$

When $x = 0$,

This eliminates the term Bx.

$$3(0)^2 + 2(0) + 3 = A(0^2 + 3) + C(0 + 1)$$

$$\Rightarrow 3 = 3A + C$$

◀ **From above working, $A = 1$.**

$$\Rightarrow 3 = 3 + C$$

$$\Rightarrow C = 0$$

Equating the coefficients of the powers of x:

$$x^2:\quad 3 = A + B \qquad [1]$$

$$x:\quad 2 = B + C \qquad [2]$$

$$\text{constants:}\quad 3 = 3A + C \qquad [3]$$

We know $A = 1$ and $C = 0$, so equation [1] or [2] can be used to find B. From [1],

$3 = A + B$

$3 = 1 + B$

$B = 2$ ◀ **Check using equation [2].**

So $\dfrac{3x^2 + 2x + 3}{(x+1)(x^2+3)} = \dfrac{1}{(x+1)} + \dfrac{2x}{(x^2+3)}$. ◀ **Check this result.**

Repeated factors in the denominator

Some algebraic fractions have denominators that contain repeated factors; for example, $\dfrac{3x+5}{(x+2)^2}$ and $\dfrac{5x+15}{(x-1)(x+4)^2}$.

What happens when fractions of this type are written in partial fractions?

Example 7

Express $\dfrac{3x+5}{(x+2)^2}$ in partial fractions.

Solution

$$\frac{3x+5}{(x+2)^2} \equiv \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

Notice that the partial fractions have denominators $(x+2)$ and $(x+2)^2$. The two numerators are assumed to be constants. Adding the algebraic fractions gives

$$\frac{3x+5}{(x+2)^2} \equiv \frac{A(x+2)+B}{(x+2)^2}$$

The two numerators can now be equated since the denominators of these two fractions are equivalent.

$$3x + 5 \equiv A(x+2) + B$$

Since this statement is true for all values of x substitutions can be made.

Alternatively, equate coefficients of x and constants.
$x: 3 = A$
constants: $5 = 2A +$
Check that this gives same solution.

When $x = -2$, $3(-2) + 5 = B$

$$\Rightarrow \quad -6 + 5 = B$$

$$B = -1$$

When $x = 0$, $3(0) + 5 = A(0 + 2) + B$

$$\Rightarrow \quad 5 = 2A + B$$

$$5 = 2A - 1$$

$$2A = 6$$

$$A = 3$$

So $$\frac{3x+5}{(x+2)^2} \equiv \frac{3}{(x+2)} + \frac{-1}{(x+2)^2} \equiv \frac{3}{(x+2)} - \frac{1}{(x+2)^2}$$

Check this by:
(i) adding together the algebraic fractions;
(ii) putting $x = 10$.

Example 8

Express $\dfrac{5x+15}{(x-1)(x+4)^2}$ in partial fractions.

Solution

Notice that the denominator has a linear factor $(x - 1)$ and a repeated factor $(x + 4)$. This means the fraction will split into three partial fractions, with denominators $(x - 1)$, $(x + 4)$ and $(x + 4)^2$.

$$\frac{5x+15}{(x-1)(x+4)^2} \equiv \frac{A}{(x-1)} + \frac{B}{(x+4)} + \frac{C}{(x+4)^2}$$

Adding together the algebraic fractions with a common denominator of $(x - 1)(x + 4)^2$,

$$\frac{5x+15}{(x-1)(x+4)^2} \equiv \frac{A(x+4)^2 + B(x-1)(x+4) + C(x-1)}{(x-1)(x+4)^2}$$

Equating the numerators,

$$5x + 15 \equiv A(x+4)^2 + B(x-1)(x+4) + C(x-1)$$

When $x = 1$, $5(1) + 15 = A(1 + 4)^2$ ◄ **Substitute suitable values of x.**

$$\Rightarrow \quad 5 + 15 = 25A$$

$$\Rightarrow \quad A = \frac{20}{25}$$

$$\Rightarrow \quad A = \frac{4}{5}$$

When $x = -4$, $5(-4) + 15 = C(-4 - 1)$

$$\Rightarrow \quad -20 + 15 = -5C$$

$$\Rightarrow \quad -5 = -5C$$

$$\Rightarrow \quad C = 1$$

To determine B, either substitute another value of x or equate coefficients of some power of x. Equating coefficients is usually the quicker method. So, equating coefficients of x^2,

$$x^2: \quad 0 = A + B$$

Try equating coefficie of x or constant terms check whether you ge the same result.

We know $A = \frac{4}{5}$, so $B = -\frac{4}{5}$.

$$\text{So } \frac{5x+15}{(x-1)(x+4)^2} \equiv \frac{\frac{4}{5}}{(x-1)} - \frac{\frac{4}{5}}{(x+4)} + \frac{1}{(x+4)^2}$$

$$\equiv \frac{4}{5(x-1)} - \frac{4}{5(x+4)} + \frac{1}{(x+4)^2}$$

Check this result by adding the algebraic fractions.

Improper fractions

These techniques can be extended to improper algebraic fractions. Remember that fractions are improper when the numerator is of a degree equal to, or higher than, the denominator. In these cases polynomial long division must be used first.

Example 9

Write the improper fraction $\frac{5x^3+15}{(x-2)(x+3)}$ in partial fractions.

Solution

Notice that the numerator is a polynomial of degree 3 and the denominator is a quadratic in factorised form; the fraction is improper.

$$(x-2)(x+3) = x^2 + x - 6$$

Now rewrite the problem in the usual long division format. The long division algorithm gives

Remember to include $0x^2$ and $0x$ to retain t column structure.

$$\begin{array}{r}
5x - 5 \\
x^2+x-6\overline{)5x^3 + 0x^2 + 0x + 15} \\
\underline{5x^3 + 5x^2 - 30x} \quad \downarrow \\
-5x^2 + 30x + 15 \\
\underline{-5x^2 - 5x + 30} \\
35x - 15
\end{array}$$

$$\text{So } \frac{5x^3+15}{(x-2)(x+3)} = 5x - 5 + \frac{35x-15}{(x-2)(x+3)}$$

The quotient is $(5x-5$ and the remainder is $35x-15$.

We can now write the proper algebraic fraction in terms of partial fractions, in the same way as before.

$$\frac{35x-15}{(x-2)(x+3)} \equiv \frac{A}{(x-2)} + \frac{B}{(x+3)}$$

Since the denominators are all linear we can use the cover-up rule.
Covering up $(x-2)$ and putting $x = 2$,

$$\frac{35x-15}{(\quad)(x+3)} = \frac{70-15}{5} = 11$$

Covering up $(x+3)$ and putting $x = -3$,

$$\frac{35x-15}{(x-2)(\quad)} = \frac{-105-15}{-5} = 24$$

So $A = 11$ and $B = 24$.

So $\dfrac{5x^3+15}{(x-2)(x+3)} \equiv 5x - 5 + \dfrac{11}{(x-2)} + \dfrac{24}{(x+3)}$

Check this result by substitution, using $x = 10$.

The partial fractions techniques can be summarised as follows:

Factors of the denominator	Example	Partial fractions
Linear	$\dfrac{5x+7}{(x+3)(x-1)}$	$\dfrac{A}{x+3}+\dfrac{B}{x-1}$
	$\dfrac{2x-3}{(x+4)(x+1)(x-5)}$	$\dfrac{A}{x+4}+\dfrac{B}{x+1}+\dfrac{C}{x-5}$
Quadratic that does not factorise	$\dfrac{3x^2+5x+6}{(x+1)(x^2+3)}$	$\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+3}$
Repeated	$\dfrac{5x+15}{(x-1)(x+4)^2}$	$\dfrac{A}{x-1}+\dfrac{B}{x+4}+\dfrac{C}{(x+4)^2}$

A useful rule to remember is that the number of constants ($A, B, C, \ldots$) needed when first writing down the partial fractions is equal to the order of the denominator of the original proper fraction. For example, $(x-1)(x+4)^2$ is of order 3, so three constants, A, B and C, are needed.

6.3 Partial Fractions
Exercise
Technique

1 Express the following in partial fractions (denominators with linear factors):

a $\dfrac{6}{(x-5)(x+1)}$ d $\dfrac{2(3x+5)}{(x-1)(x+3)}$

b $\dfrac{2x}{(x-1)(x+3)}$ e $\dfrac{7x-12}{x^2-3x+2}$ 1 b

c $\dfrac{5x+29}{(x-4)(x+3)}$ f $\dfrac{5x-17}{x^2-7x+12}$

2 Express the following in partial fractions (denominators with quadratic factors):

a $\dfrac{3x^2-2x+7}{(x-2)(x^2+1)}$ d $\dfrac{4x^2+5x+9}{(x+1)(x^2+x+4)}$

b $\dfrac{5x^2-2x+15}{x(x^2+3)}$ e $\dfrac{3x^2+20x+3}{(x-2)(x^2+3x+1)}$ 2 b

c $\dfrac{2(x^2+2x-4)}{(x-3)(x^2+2)}$ f $\dfrac{3x^2+5x+14}{(x+1)(x^2+x+4)}$

3 Express the following in partial fractions (denominators with a repeated factor):

a $\dfrac{x+2}{(x+4)^2}$ d $\dfrac{2(2x+3)}{(x+3)^2}$

b $\dfrac{x+1}{(x-2)^2}$ e $\dfrac{4x-1}{(x-1)^3}$

c $\dfrac{2x+5}{(x+5)^2}$ f $\dfrac{18x-9}{(x+2)(x-1)^2}$ 3 f

4 Express these improper algebraic fractions as partial fractions:

a $\dfrac{2x^3-x^2-18x-22}{(x-4)(x+2)}$ d $\dfrac{10x^3+20}{(x-3)(x^2+1)}$

b $\dfrac{5x^3+15}{(x-2)(x+3)}$ e $\dfrac{x^3-4x}{(x-2)(x+1)^2}$

c $\dfrac{3x^3+12}{(x-1)(x^2+2)}$ f $\dfrac{2x^3+4x^2+5x-5}{(x+1)^2}$

5 Express the following in partial fractions:

a $\dfrac{3x^2+2x-41}{(x+1)(x-3)(x-4)}$ d $\dfrac{3x^2+5x+6}{(x+1)(x^2+3)}$

b $\dfrac{8x^2-34x+32}{(x-3)(x-2)(x-1)}$ e $\dfrac{x^2+11x+4}{(x-1)(x+1)^2}$

c $\dfrac{4x^2+16x-60}{(x+1)(x^2-25)}$ f $\dfrac{4x^2+x-2}{(x-2)(x^2+x+2)}$

6.4 Curve Sketching

'Curve-sketching' means producing an outline graph that shows the general behaviour of a function or polynomial. It is not necessary to draw a table of values and plot points accurately. The sketch should show the main features.

Step ① Find where the graph crosses or intercepts the axes.
Step ② Identify any stationary points (local maxima, minima or stationary points of inflexion).
Step ③ Find out what happens for large positive and negative values of the variable (the behaviour as x tends to $+\infty$ and $-\infty$).
Step ④ Identify any values for which the function is undefined (or discontinuous).

The symbol for infinity, ∞, was introduced by John Wallis in 1655.

Example 1

Sketch the curve $y = 2x^2 + 8x + 6$.

Solution

The main features of this curve are identified separately and then a sketch is produced combining them all.

- The graph crosses the y-axis when $x = 0$. ◀ **① Intercepts.**

  $x = 0 \Rightarrow y = 2(0)^2 + 8(0) + 6 = 6$

  So the graph passes through $(0, 6)$.

  The graph crosses the x-axis when $y = 0$.

  $$y = 0 \Rightarrow 0 = 2x^2 + 8x + 6$$
  $$\Rightarrow 2(x^2 + 4x + 3) = 0$$
  $$\Rightarrow 2(x + 3)(x + 1) = 0$$
  $$\Rightarrow (x + 3) = 0 \quad \text{or} \quad (x + 1) = 0$$
  $$\Rightarrow x = -3 \quad \text{or} \quad x = -1$$

  Use factorisation to solve this quadratic equation.

  So the graph passes through $(-3, 0)$ and $(-1, 0)$.

- Stationary points occur when $\frac{dy}{dx} = 0$. ◀ **② Stationary points.**

  $$y = 2x^2 + 8x + 6 \Rightarrow \frac{dy}{dx} = 4x + 8$$

  $$\frac{dy}{dx} = 0 \Rightarrow 4x + 8 = 0 \Rightarrow x = -2$$

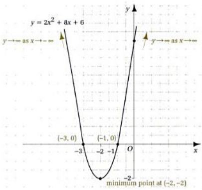
  When $x = -2$, $y = 2(-2)^2 + 8(-2) + 6 = -2$

So there is a stationary point at $(-2, -2)$.
Using the techniques from Chapter 5, verify that it is a minimum point.

Remember to look at the gradient of the curve either side of $x = -2$.

- As x gets large ($x \to +\infty$), y also gets large ($y \to +\infty$). ◀ ③ **Large x.** Similarly as $x \to -\infty$, $y \to +\infty$ because of the x^2 term.
- The function is defined for all values of x, so the graph should be a continuous curve. ◀ ④ **Continuity.**

Now combine all four findings into one graph. Verify that the graph of $y = 2x^2 + 8x + 6$ has this form using a graphical calculator.



Example 2

Sketch the curve $y = x^3 + 2x^2 - x - 2$.

Solution

The graph crosses the y-axis when $x = 0$.

- $x = 0 \Rightarrow y = (0)^3 + 2(0)^2 - (0) - 2 = -2$ ◀ ① **Intercepts.**

The graph passes through $(0, -2)$.

The graph crosses the x-axis when $y = 0$.

$y = 0 \Rightarrow x^3 + 2x^2 - x - 2 = 0$

This cubic equation can be solved using the factor theorem.

Let $P(x) = x^3 + 2x^2 - x - 2$
Check that $P(-2)$, $P(-1)$ and $P(1)$ are all zero.
This means $(x + 2)$, $(x + 1)$ and $(x - 1)$ are all factors of $P(x)$.
This means the graph crosses the x-axis at three distinct points: $(-2, 0)$, $(-1, 0)$ and $(1, 0)$.

Check that $P(x) = (x + 2)(x + 1)(x$

- Stationary points can be found from $\frac{dy}{dx} = 0$. ◀ ② **Stationary points.**

  If $y = x^3 + 2x^2 - x - 2$ then $\frac{dy}{dx} = 3x^2 + 4x - 1$

  So $\frac{dy}{dx} = 0 \Rightarrow 3x^2 + 4x - 1 = 0$

  This particular quadratic equation cannot be solved by factorising, so use 'completing the square' or the 'quadratic formula' instead.

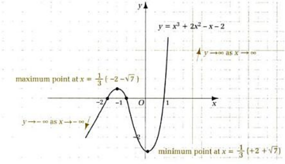
  Verify that the solutions are $x = \frac{1}{3}(-2 \pm \sqrt{7})$.

  This means there are two turning points. The approximate coordinates are $(-1.55, 0.63)$ and $(0.22, -2.11)$. Check that these turning points are a maximum and a minimum point, respectively, by looking at the values of $\frac{dy}{dx}$ on either side of each point.

- As $x \to \infty$, $y \to \infty$ because of the behaviour of the x^3 term. Similarly, as $x \to -\infty$, $y \to -\infty$. ◀ ③ **Large x.**

- The graph will be defined for all values as this polynomial can be evaluated for all values of x. ◀ ④ **Continuity.**

Combine all four findings into one graph.



Check this graph using a graphical calculator.

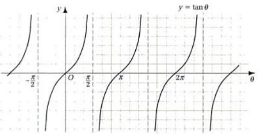
Graphs of discontinuous functions

Discontinuous functions have graphs that get closer and closer to some straight line and yet never cross it. This straight line is called an **asymptote**. This type of graph has been met already.

Look at the graph of $y = \tan\theta$ in Chapter 3.

Recall that $\tan\theta$ is undefined for a sequence of values of θ.

$$\ldots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$



What is special about these values of θ?

Recall that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

For these values of θ, $\cos\theta = 0$. The rational function that then defines $\tan\theta$ has a denominator of zero.

This observation indicates a useful technique to adopt when sketching rational functions. Identify those values that produce a zero denominator. Vertical asymptotes will be located at those values.

Example 3

Sketch the graph of $y = \dfrac{3}{(x-2)}$.

Solution

- $x = 0 \Rightarrow y = \dfrac{3}{-2} = -\dfrac{3}{2}$ ◀ ① **Intercepts.**

  The graph crosses the y-axis at $(0, -\frac{3}{2})$.

  $y = 0 \Rightarrow 0 = \dfrac{3}{(x-2)}$

  Since there is no value of x for which $y = 0$ the graph does not cross the x-axis. So the line $y = 0$ (the x-axis) must be an asymptote.

- $y = \dfrac{3}{(x-2)} \Rightarrow \dfrac{dy}{dx} = -\dfrac{3}{(x-2)^2}$ ◀ ② **Stationary points.**

  Since $\frac{dy}{dx} \neq 0$ for all values of x there are no stationary points.

- As $x \to +\infty$, $y \to 0^+$ ◀ ③ **Large x.**

  As $x \to -\infty$, $y \to 0^-$

- The function $\frac{3}{(x-2)}$ is undefined when $x - 2 = 0$ ◀ ④ **Continuity.**

  So the graph is discontinuous when $x = 2$.

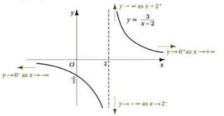
  Since $x = 2$ is an asymptote the behaviour of y either side of this value should be checked.

  As $x \to 2^+$ (from above), $x - 2 \to 0^+$, so $y = \dfrac{3}{(x-2)} \to +\infty$.

  As $x \to 2^-$ (from below), $x - 2 \to 0^-$, so $y \to -\infty$.

0^+ means $x \to 0$ fron above (positive) and 0^- means $x \to 0$ fron below (negative).

All of these features can now be combined to give the graph of $y = \dfrac{3}{(x-2)}$.



Check this result using a graphical calculator.

Graphical calculator support pack

Example 4

Sketch the graph of $y = \dfrac{x+1}{x+5}$.

Solution

Look closely at the algebraic fraction. Spot that it is improper and rewrite it in quotient/remainder form.

Verify that $y = \dfrac{x+1}{x+5}$ can be written as $y = 1 - \dfrac{4}{x+5}$.

- Putting $x = 0$ gives $y = \frac{1}{5}$. So $(0, \frac{1}{5})$ is the intercept on the y-axis. Putting $y = 0$ gives $x = -1$. So the graph crosses the x-axis at $(-1, 0)$.
- Verify that $\dfrac{dy}{dx} = \dfrac{4}{(x+5)^2}$. Since $\dfrac{dy}{dx} \neq 0$ for all values of x, there are no stationary points.
- As $x \to \pm\infty$, $y \to 1$, so $y = 1$ is a horizontal asymptote.
- The graph is not defined for $x = -5$. When $x = -5$ the denominator is zero. So $x = -5$ is a vertical asymptote.

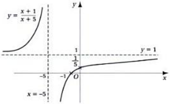
These can now be combined to give the graph.

① Intercepts.

② Stationary points.

③ Large x.

④ Continuity.



Check this result using a graphical calculator or by transforming the graph of $y = \frac{1}{x}$ using the techniques from Chapter 4.

Example 5

Sketch the graph of $y = \dfrac{x^2+2}{x-1}$.

Solution

Look closely at the algebraic fraction. Spot that it is improper.

Check that $y = \dfrac{x^2+2}{x-1}$ can be written as $y = x + 1 + \dfrac{3}{x-1}$.

The quotient is now $(x + 1)$ and contains the variable x.

Writing the equation in different forms allows you to gather relevant information more easily.

- The graph crosses the y-axis when $x = 0$ at $(0, -2)$. ◀ ① **Intercepts.**

  Confirm this result.

  The graph does not cross the x-axis. Why? If $y = 0$ then the numerator $x^2 + 2 = 0$, and this equation has no real solutions.

- If $y = x + 1 + \dfrac{3}{x-1}$, then $\dfrac{dy}{dx} = 1 - \dfrac{3}{(x-1)^2}$ ◀ ② **Stationary points.**

  For stationary values, $\frac{dy}{dx} = 0$.

  $$1 - \frac{3}{(x-1)^2} = 0 \Rightarrow (x-1)^2 = 3$$
  $$\Rightarrow \quad x - 1 = \pm\sqrt{3}$$
  $$\Rightarrow \quad x = 1 \pm \sqrt{3}$$

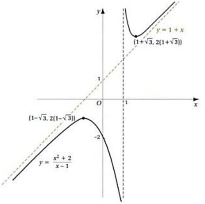
  Check that the stationary point at $x = 1 + \sqrt{3}$ is a minimum point and that the stationary point at $x = 1 - \sqrt{3}$ is a maximum point by looking at the value of the gradient of the curve on either side.

- As x becomes very large (both positive and negative), $\dfrac{3}{x-1}$ becomes very small, and tends towards zero. Therefore $y \to x + 1$ as $x \to \pm\infty$. So $y = x + 1$ is a skew, or slant asymptote.

  ③ Large x.

- The graph is not defined for $x = 1$. When $x = 1$ the original denominator is zero. So $x = 1$ is a vertical asymptote.

  ④ Continuity.



Modulus graphs

The modulus of a function, $|f(x)|$, was defined in Chapter 4. To sketch graphs of the modulus of a function, first sketch the graph of the function $f(x)$, and then reflect in the x-axis all parts of the graph for which the y-coordinate is negative.

Example 6

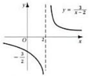
Sketch the graphs of:

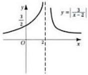
a $y = \left|\dfrac{3}{x-2}\right|$ **b** $y = \left|\dfrac{x+1}{x+5}\right|$ **c** $y = \left|\dfrac{x^2+2}{x-1}\right|$.

Solution

Compare these with the graphs of the functions found in Examples 3–5. All that is required now is to reflect the parts of the graph that are below the x-axis (where y-coordinates are zero) in the x-axis.

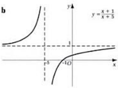
a

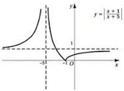




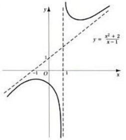
Check these results using a graphical calculator.

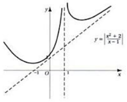
b





c





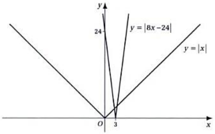
Sketching a modulus graph is a very useful technique when we have to solve equations or inequalities that involve the modulus function.

Example 7

Solve the equation $|x| = |8x - 24|$. Hence, or otherwise, solve $|x| \leq |8x - 24|$.

Solution

To solve the equation first sketch the graphs of $y = |x|$ and $y = |8x - 24|$ on the same axes.
Notice that we get two distinct V-shaped graphs. The graph of $y = |8x - 24|$ is steeper and touches the x-axis at $x = 3$.



The points of intersection of the two graphs are the solutions to $|x| = |8x - 24|$. These can be found algebraically by solving two sets of equations:

$$x = 8x - 24$$

and $$x = -(8x - 24)$$

The first equation corresponds to the point where $y = x$ intersects $y = 8x - 24$. The second equation gives the solution created by the reflection in the x-axis. This corresponds to the point where $y = x$ intersects $y = -(8x - 24)$.
Solving these equations gives

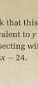
$$x = 8x - 24 \quad \Rightarrow \quad 7x = 24$$
$$x = \tfrac{24}{7}$$
$$x = -(8x - 24) \quad \Rightarrow \quad 9x = 24$$
$$x = \tfrac{24}{9}$$

To solve the inequality $|x| \leq |8x - 24|$, use the graph. Notice that the graph of $y = |x|$ is below the graph of $y = |8x - 24|$ when $x \leq \frac{24}{9}$ or $x \geq \frac{24}{7}$. So $|x| \leq |8x - 24|$ when $x \leq \frac{24}{9}$ and $x \geq \frac{24}{7}$.
When we need to solve inequalities that involve the modulus function it is often useful to check the graphs. We can usually identify where the required condition is, geometrically, by finding where a graph is either above or below another graph or axis.

Check that this is equivalent to $y = -x$ intersecting with $y = 8x - 24$.

Check that $x = \frac{24}{7}$ an[d] $x = \frac{24}{9}$ satisfy the equ[ation] $|x| = |8x - 24|$.

Verify this result on [a] graphical calculator.



6.4 Curve Sketching
Exercise

Technique

1 Sketch the graphs of the following continuous functions:

a $f(x) = 2x^3 + 7x^2 - 5x - 4$ b $h(x) = |x^2 - 6x - 7|$
c $g(x) = x^3 - 19x + 30$ d $h(x) = |2x^3 + 15x^2 + 31x + 12|$

2 Sketch the graphs of the following rational functions:

a $g(x) = \dfrac{3}{x-4}$ b $f(x) = \dfrac{2}{1-x}$
c $g(x) = \dfrac{1}{3-4x}$ d $h(x) = \left|\dfrac{1}{2x+3}\right|$

3 Sketch the graphs of the following rational functions:

a $f(x) = \dfrac{x+1}{x-3}$ b $g(x) = \dfrac{x+3}{x+4}$
c $h(x) = \dfrac{x-3}{x-2}$ d $h(x) = \left|\dfrac{5x+1}{3x-2}\right|$

4 Solve the following:
a $|x| = |5x - 40|$ b $|x| = |12 - 3x|$ c $|x - 2| = |6 - 2x|$
d $|x| \leq |8x - 40|$ e $|x| \geq |1 - 5x|$ f $|2x + 1| \geq |5 - 3x|$

Contextual

1 Given that $f(x) = 3x^3 - 7x^2 - 7x + 3$:

a factorise $f(x)$ and then sketch its graph
b on a separate diagram sketch the graph of $|f(x)|$.

2 Given that $f(x) = 2x + 1$ and $g(x) = x + 2$, find the range of values of x for which:

a $f(x) > 0$ b $g(x) > 0$ c $\dfrac{f(x)}{g(x)} > 0$

Hint: Sketch the appropriate graphs first.

3 Sketch the graph of $y = \dfrac{4x+1}{2x-5}$. Hence or otherwise find the range of values of x for which $y < 0$

4 On separate diagrams sketch the graphs of:

a $y = \dfrac{2x-6}{x+8}$ b $y = \left|\dfrac{2x-6}{x+8}\right|$

4 a, b

Consolidation

Exercise A

1 Use the factor theorem to find a linear factor of $x^3 - x - 6$. Hence express $x^3 - x - 6$ as the product of a linear factor and a quadratic factor.

(OCSEB)

2 It is given that $g(x) = (2x - 1)(x + 2)(x - 3)$.

a Express $g(x)$ in the form $Ax^3 + Bx^2 + Cx + D$, giving the values of the constants A, B, C and D.

b Find the value of the constant a, given that $(x + 3)$ is a factor of $g(x) + ax$.

c Express $\frac{x - 3}{g(x)}$ in partial fractions.

d Solve the inequality $g(x) > 3x(x + 2)(x - 3)$.

(UCLES)

3 **a** When divided by $(x - 3)$ the expression $x^3 + ax^2 - (a^2 - 3a + 17)x$ gives a remainder of 3. Find the value of a.

b Find the three linear factors of $3x^3 - x^2 - 75x + 25$.

4 The polynomial $p(x) = x^3 + cx^2 + 7x + d$ has a factor of $(x + 2)$, and leaves a remainder of 3 when divided by $(x - 1)$.

a Determine the value of each of the constants c and d.

b Find the exact values of the three roots of the equation $p(x) = 0$.

(AEB)

5 Given that $\frac{(x - 1)^3}{x^2} = Ax + B + \frac{C}{x} + \frac{D}{x^2}$, $x \neq 0$, find the values of A, B, C and D.

(OCSEB)

6 Given that $y = \frac{2x + 1}{x(2x - 1)^2}$, express y in the form $\frac{A}{x} + \frac{B}{(2x - 1)} + \frac{C}{(2x - 1)^2}$.

(NEAB)

7 The cubic polynomial $x^3 - 2x^2 - x - 6$ is denoted by $f(x)$. Show that $(x - 3)$ is a factor of $f(x)$. Factorise $f(x)$. Hence find the number of real roots of the equation $f(x) = 0$, justifying your answer.

Hence write down the number of points of intersection of the graphs with equations $y = x^2 - 2x - 1$ and $y = \frac{6}{x}$, justifying your answer.

(UCLES)

8 If $p(x) = 2x^3 - 5x^2 - 28x + 15$:

a factorise $p(x)$ **b** solve $p(x) = 0$
c sketch the graph of $p(x)$ **d** solve $p(x) \geq 0$.

9 Sketch the curve $y = \dfrac{x+3}{x-5}$. Now find the values of x for which $y > 0$.

10 Given $f(x) = \dfrac{2x+1}{x+2}$:

a sketch the graph of $f(x)$
b on a separate diagram sketch the graph of $|f(x)|$.

Exercise B

1 **a** When the cubic expression $x^3 + ax^2 - (2a^2 + 12)x + (7a + 10)$ is divided by $(x - 1)$ the remainder is 7. Find a.
b Find the three linear factors of $x^3 + 2x^2 - 20x + 24$. *(WJEC)*

2 A polynomial $f(x)$ can be expressed in the form $p(x+2)^4 + q$ where p and q are constants. When $f(x)$ is divided by $(x - 1)$ the remainder is 40 and when $f(x)$ is divided by $(x + 3)$ the remainder is zero. Find the values of p and q. *(NEAB)*

3 Given $f(x) = (3x - 2)(x - 1)(x + 1)$:

a express $f(x)$ in the form $Ax^3 + Bx^2 + Cx + D$
b express $\dfrac{x+1}{f(x)}$ in partial fractions
c solve the inequality $f(x) \geq 2x(x - 1)(x + 1)$.

4 The cubic polynomial $f(x) = 5x^3 + px^2 - 11x + q$ has a remainder of -12 when divided by $(x - 1)$ and also has an exact factor $(x - 3)$.

a Find the values of p and q.
b Express the cubic polynomial as the product of three linear factors.
c With the aid of a sketch, solve $f(x) < 0$.

5 A curve has equation $y = \dfrac{3x+4}{(x-2)(2x+1)}$.

a Express $\dfrac{3x+4}{(x-2)(2x+1)}$ in partial fractions.
b Show that $\dfrac{dy}{dx} = \dfrac{2}{(2x+1)^2} - \dfrac{2}{(x-2)^2}$ and hence, or otherwise, show that the curve has a turning point when $x = -3$. Determine the value of x at the other stationary point of the curve. *(AEB)*

Applications and Activities

Parametric equations

Sometimes, instead of having an equation relating x and y directly, $y = f(x)$, x and y are written in terms of a third variable, called a **parameter**. Then x and y are represented by two separate functions, $x = g(t)$ and $y = h(t)$, called **parametric equations**. Eliminating the parameter, or variable, reduces the two parametric equations into a Cartesian form with two variables.

Parametric equations are a useful technique to use with graphical calculators, when it is difficult to write y as a function of x. Use a graphical calculator in the parametric mode to help you with the following activities.

1 Sketch the graphs of the curves given parametrically by the following and describe the Cartesian form of these curves:

a $x = \cos\theta$ and $y = \sin\theta$

b $x = 3 + \cos\theta$ and $y = 4 + \sin\theta$

c $x = -3 + \cos\theta$ and $y = 4 + \sin\theta$

d $x = -3 + \cos\theta$ and $y = -4 + \sin\theta$

e $x = 3 + \cos\theta$ and $y = -4 + \sin\theta$

2 Predict, and then check, the behaviour of the curve given parametrically by $x = 2\cos\theta$ and $y = 2\sin\theta$. What do you expect the curve given parametrically by $x = 5 + 2\cos\theta$ and $y = 1 + 2\sin\theta$ will look like?

3 Investigate other parametric coordinates.

Summary

- A polynomial can be divided by a linear factor and the result is checked using either multiplication or substitution.
- The **remainder theorem** says that when a polynomial $P(x)$ is divided by a linear factor $(x - a)$ then the remainder is $P(a)$.
- The **factor theorem** says that if $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.
- The **factor theorem** can be used to factorise cubic and higher order polynomial expressions.
- **Partial fractions** can be classified into three categories: those with linear factors in their denominators; those with quadratic factors in the denominator; and those with repeated factors in the denominator.
- The **cover-up rule** uses the fact that some values of x create a zero in the denominator to calculate the required numerators in the partial fractions.
- To sketch the graphs of continuous and discontinuous functions, check:
  ① where the graph crosses the axes
  ② the location and nature of any stationary points
  ③ the behaviour as $x \to \pm\infty$
  ④ where the function is undefined.
- The graph can be discontinuous at a point. This is shown by a line called an **asymptote**.
  - A *vertical* asymptote can be found when the denominator of a rational function is zero.
  - A *horizontal* asymptote can be found when the function tends towards a fixed value for large positive and negative values of x.
  - A *skew* asymptote can be found by writing an improper algebraic fraction in its quotient–remainder form.
- Produce a sketch of the modulus of a function, $|f(x)|$, by reflecting in the x-axis all parts of the graph for which the function is negative.

7 Exponentials and Logarithms

What you need to know

- How to write numbers in standard index form.
- The laws of indices.
- How to evaluate negative and fractional indices.
- How to use the $x^{1/y}$ or $\sqrt[x]{\ }$ function key on your calculator.

Review

1 Write the following numbers in standard (index) form:

a 93 000 000 **b** 0.006 25
c ten million **d** three hundredths

2 Simplify the following:

a $a^7 \times a^5$ **b** $a^6 \div a^2$ **c** $(m^6)^3$
d m^0, where $m \neq 0$ **e** $a^m \times a^n$ **f** $\dfrac{a^m}{a^n}$

3 Without using a calculator, work out:

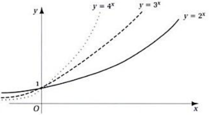
a $16^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** 4^{-3}
d $(\frac{1}{4})^{\frac{1}{2}}$ **e** $64^{\frac{2}{3}}$ **f** $125^{-\frac{1}{3}}$

4 Use your calculator to find the values of the following, correct to three significant figures:

a $10^{\frac{1}{3}}$ **b** $(0.03)^{\frac{2}{3}}$
c $(1.2)^{-3}$ **d** $(-200)^{-\frac{1}{3}}$

7.1 The Exponential Function

Any function of the form a^x, where a is a positive constant, is known as an **exponential** (or power) **function**. The variable of the function is x. Examples of exponential functions include 2^x, 3^{5x}, 5^x and 10^{x+3}. The graphs of these functions all have the same basic shape.



Since $a^0 = 1$, all graphs of the form $y = a^x$ cross the y-axis at $(0, 1)$.

- The graphs are defined for all values of x.
- The graphs are always above the x-axis.
- The x-axis (the line $y = 0$) is an asymptote, and each graph tends to infinity quickly as x gets large.

Exponential functions are always positive.

Exponential functions can be used to model growth and decay. To model decay we find that the exponent needs to be negative. Graphs of exponential functions with negative exponents are the same shape as those with positive exponents, but reflected in the line $x = 0$ (or the y-axis).

Exponential functions are used to predict population growth, describe radioactive decay and determine compound interest.

Population growth

The world's human population is growing by about 3% each year. Given that the population at the end of 1989 was estimated at 4.5 billion (4.5×10^9), we can predict population figures for future years using an exponential graph.

Multiplying by 1.03 increases by 3%.

Year end	1989	1990	1991	1992	1993
Population in billions	4.5	4.5×1.03	$4.5 \times (1.03)^2$	$4.5 \times (1.03)^3$	$4.5 \times (1.03)^4$

Graphical calculator support pack

Complete a table of values or use a graphical calculator to draw the graph of $y = 4.5 \times 1.03^x$. Notice that the curve crosses the y-axis at 4.5 (representing 4.5 billion in 1989, our initial year). Using a trace and zoom function we can find when the population doubles ($y = 2 \times 4.5 = 9$).

We find that $y = 9.0$ when $x = 23.449772$. So $x \approx 24$.

This means we can predict that the world's human population will have doubled in size from its 1989 estimate by the end of the year 2013 ($1989 + 24$ years).

Radioactive decay

Radioactive substances have 'half-lives' that are determined by the time it takes the radioactivity to halve. Radon-219, an isotope of the element radon, has a half-life of approximately four seconds. This means that if at some time $t = 0$ we have 10 000 Radon-219 nuclei, then four seconds later ($t = 4$) there will be only 5000.

The exponential model for radioactive decay will have a negative exponent (power). To find it, first tabulate some values.

Time after sample is identified	0	4	8	12	16
Number of nuclei remaining	10 000	$10\,000 \times (\frac{1}{2})$ $= 5000$	$10\,000 \times (\frac{1}{2})^2$ $= 2500$	$10\,000 \times (\frac{1}{2})^3$ $= 1250$	$10\,000 \times (\frac{1}{2})^4$ $= 625$

Notice that every four seconds the power increases by 1. If $N(t)$ is the number of atoms at time t, then

$$
\begin{aligned}
N(t) &= 10\,000 \times (\tfrac{1}{2})^{\frac{t}{4}} \\
&= 10\,000 \times (2^{-1})^{\frac{t}{4}} \\
&= 10\,000 \times 2^{-\frac{t}{4}}
\end{aligned}
$$

This model now allows us to calculate the number of nuclei of Radon-219 present in the sample at any time after timing commenced ($t = 0$).

Compound interest

Banks and building societies add the interest to certain accounts at the end of a year. In the following year interest is also calculated on this interest. This method of calculation is known as 'compound interest'. Consider investing £500. What would give the best return, 10% per year or 1% every $\frac{1}{10}$ year? Compare the balance at the start of each year.

Year	1	2	3	4	5
10% per year (£)	500	550	605	665.50	732.05
1% every $\frac{1}{10}$ year (£)	500	552.31	610.10	673.92	744.43

Trace and zoom
The 'Trace' facility o[f] graphical calculator [is] described on p. 34. T[he] 'Zoom' facility enabl[es] you to zoom in on a region, or section, of [a] plotted graph. Most calculators will zoom [in] on the graph at the point traced along to and then redraw the graph with a preset scale factor.

Remember the laws [of] indices.

Clearly the second option gives a better return. But how can these figures be calculated quickly?
In the first account £500 can be multiplied by $(1.1)^n$ where n is the number of interest paying years. In the second account £500 can be multiplied by $(1.01)^n$ where n is the number of $\frac{1}{10}$ years for which interest is paid. Thus £500 × $(1.01)^{10}$ = £552.31.
The idea of reducing the interest rate but increasing the number of times it is paid is nothing new. Today interest can be paid daily (so in a year, n would be 365) but the rate will be only a fraction of 1%.
Now look at the following pattern:

$$(1+0.1)^{10} = 2.593\,742\,46$$
$$(1+0.01)^{100} = 2.704\,813\,829$$
$$(1+0.001)^{1000} = 2.716\,923\,932$$
$$(1+0.0001)^{10\,000} = 2.718\,145\,927$$

The left-hand column is generating the pattern $(1+\frac{1}{n})^n$ for powers of 10. The figures on the RHS are converging to a limit that begins 2.718.... This number is known as 'e' and is the **limit** of this **sequence**.

This can be written mathematically as $\mathrm{e} = \underset{n\to\infty}{\text{limit}}\left(1+\frac{1}{n}\right)^n$ or more usually as $\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

The exponential function as a series

The number e can also be defined as $\mathrm{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$.

The function e^x is often referred to as *the* **exponential function**. It can be defined, in a similar way to e, as:

$$\mathrm{e}^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Alternative notation you may come across for e^x is $\exp x$ or $\exp(x)$. Do not confuse this with the EXP key on a scientific calculator.

If we differentiate this infinite power series term by term we get a remarkable result.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \mathrm{e}^x$$

That is, the gradient function equals the function itself for all real values. It is this result that makes the exponential function, e^x, unique.

7.1 The Exponential Function
Exercise
Technique

1 Using a calculator find, correct to three significant figures, the values of:

a e^2 **b** e^{-1} **c** $e^{-1} - e^{-3}$ **d** $e^{\frac{1}{2}}$

2 Use a graphical calculator to sketch the graphs of the following. In each case describe the transformation that maps $y = e^x$ onto each of these functions:

a $y = e^{(x+1)}$ **b** $y = e^x + 1$ **c** $y = e^{(x-2)}$ **d** $y = e^x - 2$

2 a, d

3 Draw the graph of $y = 2^x$. From your graph find, as accurately as you can, the value of:

a $2^{1.5}$ **b** $2^{0.3}$ **c** $2^{-0.5}$ **d** $2^{-3.5}$

4 Using a calculator, sketch the graph of $y = e^x$. Use the trace and zoom functions to find values of x correct to three significant figures for which:

a $e^x = 6$ **b** $e^x = 8.12$ **c** $e^x = 0.5$ **d** $e^x = \pi$

Contextual

1 The mass of a colony of bacteria, measured in grams, doubles each day and is given by the formula $M(t) = 7.0 \times 2^t$. Find the initial mass of the bacteria and the number of days needed for the total mass of the bacteria to exceed 500 g.

2 The decay of a radioactive isotope can be modelled by $M(t) = M_0\,e^{-kt}$, where $M(t)$ is the remaining mass after t days. If a given isotope has a half-life of 10 days and after this time 60 g remain, find:

a how much isotope was present initially (the value of M_0).
b the value of the decay constant k.
c how much isotope has remained after 30 days.
d the number of days required for 80% of the isotope to have decayed.

3 Engine oil, at temperature T°C cools down according to the model $T = 60\,e^{-kt} + 10$, where t is the time in minutes from the moment the engine was switched off.

a What is the initial temperature of the oil when the engine is first switched off?
b If the oil cools to 32°C after three minutes find the value of k.
c How long will it take for the oil to cool to a temperature of 15°C?

4 The cost of living is increasing by 4% per year. By how much does the cost of living increase in:

a a month **b** six months **c** five years?

How long will it take for the cost of living to double from its present value?

5 A car bought for £12 000 depreciates at 20% per year. After t years it is worth £x.

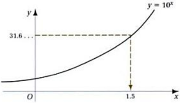
a Sketch a graph of x against t.
b Use your graph to calculate:
  i when the car is worth half its original value;
  ii when its value first falls below £1000.

7.2 Logarithms

When comparing numbers written in standard form, first look at the index, (exponent or power) of the number ten. This gives an indication of the magnitude of the number.
For example, if the index is 6 then the number will be in the millions because

$$10^6 = 1\,000\,000 = 1 \text{ million}$$

Now consider numbers between 10 and 100. Since $10 = 10^1$ and $100 = 10^2$ we would expect the index for these numbers to be between 1 and 2. What about 1.5? What number is equivalent to $10^{1.5}$? The answer is not 50. Using the 'x^y' function key on a calculator you should find that $10^{1.5} = 31.622\,777$ (6 d.p.).
The number 1.5 is called the **logarithm** of 31.622 777 in base 10. If you now try the 'log' function key on the calculator notice that $\log(31.622\,777) = 1.5$



Mathematicians generalise this relationship as follows,

If $y = a^x$ then $x = \log_a y$

and x is known as the logarithm of y in base a.

Since $31.622\,777 = 10^{1.5}$ then $1.5 = \log_{10} 31.622\,777$

Logarithms can be in bases other than 10.

Although invente
Europe by John
Napier in the sixteen
century, there is plen
evidence to show tha
logarithms were know
Asian mathematicians
the period around the
eighth century. The Ja
may have developed
the idea of logs witho
using them for any
practical purpose.

Example

For each of the following, write down an expression for a logarithm in a suitable base:

a $81 = 3^4$ **b** $8 = 2^3$

c $100 = 10^2$ **d** $\dfrac{1}{32} = \dfrac{1}{2^5}$

Henry Briggs
(1561–1630)
Logarithms to ba
10 are known as
'common logs' c
'Briggs logs' in hor
of Briggs, who
calculated the fir
table of commo
logarithms.

Solution

a $81 = 3^4$, so $4 = \log_3 81$
b $8 = 2^3$, so $3 = \log_2 8$
c $100 = 10^2$, so $2 = \log_{10} 100$
d $\frac{1}{32} = \frac{1}{2^5} = 2^{-5}$, so $-5 = \log_2(\frac{1}{32})$

Recall that $a^{-n} = \frac{1}{a^n}$.

The logarithm of a number is the power to which the base must be raised in order to equal that number. So $\log_{10} 100$ is 2 because the base, 10, must be raised to power 2 to equal 100.

The most common bases used for logarithms are 10 and e. In value, $e \approx 2.718\,2818$ and logarithms taken to this base are called **natural logarithms**.

This value can be found as your calculator by evaluating e^1.

To distinguish between these types of logarithms the following notation is used.

$\log_{10} x$ is written $\log x$ or $\lg x$

$\log_e x$ is written $\ln x$

Calculators have two different function keys for logarithms, 'LN' and 'LOG', with inverse functions e^x and 10^x respectively.
The relationship between e^x and $\ln x$ is very important.

Check that $\ln(e^x) = x$ and $e^{\ln x} = x$.

If $y = e^x$ then $x = \ln y$.

The use of e as a base for logarithms is by no means accidental. The number e has many applications in mathematics.

Transcendental numbers

A number that cannot satisfy an algebraic equation with integer coefficients is called a **transcendental number**. Many irrationals are not transcendental. For example, $\sqrt{2}$ satisfies $x^2 = 2$ or $x^2 - 2 = 0$, which are equations with integer coefficients. Both e and π are transcendental.

In 1966, Alan Baker, a British mathematician, extended French mathematician Joseph Liouville's proof of the existence of transcendental numbers.

7.2 Logarithms
Exercise
Technique

1 Without using a calculator, find the values of:

a $\log 1000$
b $\log 0.01$
c $\log 10\,000\,000$
d $\log 0.0001$

2 Find the values of the following, to three significant figures:

a $\log 50$
b $\log 2$
c $\log(\frac{1}{3})$
d $\log(\frac{2}{3})$

3 Without using a calculator find the exact values of x in:

a $x = \log_2 64$
b $x = \log_3 27$
c $x = \log_2(\frac{1}{2})$
d $x = \log_3(\frac{1}{81})$

4 Find the value of the base, a, of the following logarithms:

a $3 = \log_a 125$
b $\frac{1}{2} = \log_a 6$
c $-2 = \log_a(\frac{1}{25})$
d $\frac{1}{2} = \log_a 25$

5 Find the values of the following, to three significant figures:

a $\ln 2$
b $\ln 50$
c $\ln 100$
d $\ln 25$

6 Evaluate the following without using a calculator:

a $\ln e^2$
b $\ln \sqrt{e}$
c $\ln\left(\frac{1}{\sqrt[3]{e}}\right)$
d $\ln\left(\frac{1}{\sqrt{e^3}}\right)$

7 Copy and complete the following table:

Exponent form	Logarithmic form
$2^4 = 16$	$\log_2 16 = 4$
$2^{-1} = \frac{1}{2}$	$\log_2(\frac{1}{2}) = -1$
$2^5 = 32$	
	$\log_2 64 = 6$
$3^2 = 9$	
$3^{-2} = \frac{1}{9}$	
$4^{\frac{1}{2}} = 2$	

Contextual

1 The Richter scale measures the magnitude of earthquakes using logarithms. The Richter value is only part of the number describing an earthquake's strength. If it has a Richter value of 6, the magnitude of the earthquake is 10^6. Many people believe an earthquake with a Richter value of 8 is twice as strong as one with a value of 4. How do the two actually compare in intensity?

2 The human ear responds to the ratio of the powers, measured in watts, when the power of sound increases. This ratio is measured in bels (B) but in practice the bel is too large and the decibel (dB) is used. This is calculated using the rule

$$\text{change in dB} = 10\log_{10}\left(\frac{\text{new power}}{\text{old power}}\right)$$

a Explain why an increase in power output from 100 W to 200 W is only 3 dB.

b Approximately how many times will the power level emitted by a source of sound have increased if the sound level increases by 10 dB?

c If the volume control on a personal stereo is turned down so that the power output from the loudspeaker changes from 500 mW (milliwatts) to 100 mW, what is the corresponding fall in sound level in dB?

3 In chemistry the acidity or alkalinity of a solution is measured by its pH factor, defined by $\text{pH} = -\log_{10}[\text{H}^+]$. In this rule $[\text{H}^+]$ is a measure of the quantity of hydrogen ions present in the solution. A pH value of 7 indicates a neutral solution for which $[\text{H}^+] = 10^{-7}$ moles per litre. If the pH value is greater than 7 the solution is alkaline. If the pH value is less than 7 the solution is acidic.

a Calculate the pH value for a hair shampoo of strength 2.5×10^{-9} moles per litre.

b An acid X, has a pH of 5.0, and an acid, Y, a pH of 2.5. How many times more concentrated than acid X is acid Y?

7.3 Laws of Logarithms

If $y = a^x$ then $x = \log_a y$.

Remember that these two mathematical statements are identical and interchangeable. Now suppose that there are positive numbers c and d such that $c = m^p$ and $d = m^q$ for some $m > 0$.

Then $p = \log_m c$ and $q = \log_m d$

and $c \times d = m^p \times m^q = m^{(p+q)}$

Remember the laws of indices.

In logarithm form this can be written,

$$\log_m(c \times d) = p + q = \log_m c + \log_m d$$

This result is true for any two positive numbers c and d and for any suitable base m. Since it is true for any base,

John Napier
(1550–1617)
Napier introduced the u of logarithms as a breakthrough in simplify computation. It reduce multiplication and divis to addition and subtract This was the principle which the slide rule, forerunner to the calcula was based.

$\log ab = \log a + \log b$ **Property 1** ◀ Learn this result.

Example 1

Given $\log 2 = 0.301$ and $\log 6 = 0.778$ find $\log 12$.

Solution

$$\begin{aligned}\log 12 &= \log(2 \times 6)\\ &= \log 2 + \log 6\\ &= 0.301 + 0.778\\ &= 1.079\end{aligned}$$

Check this using you calculator.

Now return to property 1 and put $a = b$. This gives the following result:

$$\log a^2 = \log a + \log a = 2\log a$$

This can be generalised to

$\log a^n = n \log a$ **Property 2** ◀ Learn this result.

This is valid for all values of n, and not just integers.

Example 2

Given $\log 6 = 0.778$ find $\log 36$.

Solution

$$\begin{aligned}\log 36 &= \log 6^2\\ &= 2\log 6\\ &= 2\times 0.778 = 1.556\end{aligned}$$

Check this using your calculator.

Now return to $c = m^p$ and $d = m^q$ with $m > 0$ and try division instead of multiplication.

$$\frac{c}{d} = \frac{m^p}{m^q} = m^{p-q}$$

Remember the laws of indices.

In logarithm form this can be written,

$$\log_m\left(\frac{c}{d}\right) = p - q = \log_m c - \log_m d$$

This can be generalised to

$$\log\left(\frac{a}{b}\right) = \log a - \log b$$ **Property 3** ◀ **Learn this result.**

Example 3

Using the results from Examples 1 and 2, evaluate $\log 3$.

Solution

$$\begin{aligned}\log 3 &= \log\left(\frac{36}{12}\right)\\ &= \log 36 - \log 12\\ &= 1.556 - 1.079 = 0.477\end{aligned}$$

Check also that $\log 3 = \log(\frac{6}{2}) = \log 6 - \log 2$

There are two other logarithm properties worth noting, based on the indices statements $a^0 = 1$ and $a^1 = a$.

$\log_a 1 = 0$ **Property 4**

◀ **Learn these results.**

$\log_a a = 1$ **Property 5**

These five properties can be used to simplify expressions containing logarithms and to solve logarithmic equations.

The slide rule, a calculating device commonly used right up to the development of the electronic calculator, was invented by William Oughtred in 1622. The slide rule used a logarithmic scale.

Example 4

Write the following as single logarithms:

a $\log 8 - \log 6 + \log 9$ **b** $2\log a + 3\log b - \log c$

Solution

a $\log 8 - \log 6 + \log 9 = \log(\frac{8}{6}) + \log 9$ — Using property 3.

$= \log\left(\frac{8 \times 9}{6}\right)$ — Using property 1.

$= \log 12$

b $2\log a + 3\log b - \log c = \log a^2 + \log b^3 - \log c$ — Using property 2.

$= \log(a^2b^3) - \log c$ — Using property 1.

$= \log\left(\frac{a^2b^3}{c}\right)$ — Using property 3.

Some logarithmic equations are a mixture of logarithms and ordinary numbers. When that is the case, try collecting together the logarithmic terms so that the logarithm properties can be used to combine them into a single logarithmic term.

Example 5

Given that $1 + \log_a(7x - 2a) = 2\log_a x + \log_a 3$, find, in terms of a, the possible values of x.

Solution

$$1 + \log_a(7x - 2a) = 2\log_a x + \log_a 3$$

$$\Rightarrow \quad 1 = 2\log_a x + \log_a 3 - \log_a(7x - 2a)$$

$$\Rightarrow \quad 1 = \log_a x^2 + \log_a 3 - \log_a(7x - 2a)$$

Using property 2.

$$\Rightarrow \quad 1 = \log_a(3x^2) - \log_a(7x - 2a)$$

Using property 1.

$$\Rightarrow \quad 1 = \log_a \frac{3x^2}{7x - 2a}$$

Using property 3.

But $\log_a a = 1$, so

By property 5.

$$\frac{3x^2}{7x - 2a} = a$$

$$\Rightarrow \quad 3x^2 = a(7x - 2a)$$

$$\Rightarrow \quad 3x^2 = 7ax - 2a^2$$

$$\Rightarrow \quad 3x^2 - 7ax + 2a^2 = 0$$

Using PAFF:

P: $3 \times 2a^2 = 6a^2$

A: coefficient of $x = -7a$

F: $-6a$ and $-a$.

F: Replace $-7ax$ with $-6ax - ax$, and factorise.

Factorising,

$$(3x - a)(x - 2a) = 0$$

So $x = \frac{a}{3}$ or $x = 2a$

7.3 Laws of Logarithms

Exercise

Technique

1 Write the following as single logarithms:

a	$\ln 7 + \ln 2$	**d**	$\log 12 - \log 6$
b	$\log 2 + \log 3 + \log 4$	**e**	$\ln 6 + \ln 10 - \ln 5$
c	$2 \ln 3$	**f**	$\log 1 - \log 3$

1 e

2 Given that $\log 2 = 0.301$ and $\log 3 = 0.477$ find, without using a calculator:

a	$\log 4$	**b**	$\log 27$	**c**	$\log 18$
d	$\log 36$	**e**	$\log 144$	**f**	$\log 1\frac{1}{2}$

3 Simplify:

a	$\log x + 2\log y + \log z$	**d**	$\ln x^2 - \frac{1}{2}\ln x$
b	$3\ln x - 2\ln y$	**e**	$3\log x + 2\log z$
c	$\log a - 2\log b - 3\log c$	**f**	$\ln x - 2\ln y + \ln z$

4 Express, in terms of $\ln x$, $\ln y$ and $\ln z$:

a	$\ln(xyz)$	**b**	$\ln\left(\dfrac{x^3y^2}{z}\right)$	**c**	$\ln\dfrac{x\sqrt{y}}{z}$
d	$\ln\sqrt[3]{xy}$	**e**	$\ln\sqrt{\dfrac{xy}{z}}$	**f**	$\ln x\sqrt{yz^3}$

5 Solve the following logarithmic equations:

a	$\log_x 9 = \frac{1}{2}$	**b**	$\log_x 16 = -4$
c	$\ln x^3 - \ln x + \ln\sqrt{x} = 3$	**d**	$\log_x 24 + \log_x 9 + 3 = 3\log_x 4$

Contextual

1 Explain why $xy = 100$ and $\log_{10} x + \log_{10} y = 2$ are equivalent, interchangeable statements.

2 On the same axes sketch the graphs of $y = \ln x$ and $y = \ln(\frac{1}{x})$. Using the laws of logarithms suggest a possible transformation for these graphs. Describe an inverse function for $y = \ln(\frac{1}{x})$.

3
a Show that $\log_2 12 = \log_4 144$
b Evaluate $\log_2 12 - \log_4 9$ without using a calculator.

4 By taking natural logarithms, or otherwise, find $g(x)$ such that $x^x = e^{xg(x)}$ for all values of x.

5 Given that $\log_2 x + 2\log_4 y = 4$, explain why $xy = 16$.

7.4 Solving $a^x = b$

Consider the equation $3^x = 20$. This can be solved by trial and improvement using the **power function** key 'x^y' on any scientific calculator, although this method can be quite time consuming. A more concise way of solving **exponential equations** like these is to use the properties of logarithms. The method is based on the principle that if two sides of an equation are equal then the logarithm of one side must be equal to the logarithm of the other.

Example 1

Solve $3^x = 20$.

Either $\log_{10}$ or ln can be used; both bases will give the same answer.

Solution

Taking natural logarithms of both sides,

$$\ln 3^x = \ln 20$$

$$x \ln 3 = \ln 20 \quad \blacktriangleleft \text{ Using property 2.}$$

$$x = \frac{\ln 20}{\ln 3}$$

$$= \frac{2.996}{1.099} = 2.73 \text{ (3 s.f.)}$$

Check that $\log_{10}$ also gives $x = 2.73$.

The same principle can be used when the powers become more complicated.

Example 2

Solve $4^{(3x+1)} = 79$.

Solution

Taking logarithms of both sides,

Use either $\log_{10}$ or ln.

$$\log_{10} 4^{(3x+1)} = \log_{10} 79$$

$$(3x + 1)\log_{10} 4 = \log_{10} 79$$

$$3x + 1 = \frac{\log_{10} 79}{\log_{10} 4}$$

$$x = \frac{1}{3}\left(\frac{\log_{10} 79}{\log_{10} 4} - 1\right)$$

$$= \frac{1}{3}\left(\frac{1.8976}{0.6021} - 1\right) = 0.717 \text{ (3 s.f.)}$$

Check that $4^{(3 \times 0.717+}$ is close to 79. Notice rounding error.

This technique provides a useful tool for solving more complicated exponential equations where one power is a multiple of another.

Example 3

Solve the equation $e^{4x} + e^{2x} - 6 = 0$.

Solution

Notice that $4x = 2(2x)$, so this equation could be written

$$(e^{2x})^2 + e^{2x} - 6 = 0$$

Remember the laws of indices.

Now substitute $y = e^{2x}$. This transforms the exponential equation into a quadratic equation.

$$y^2 + y - 6 = 0$$

$$\Rightarrow \quad (y+3)(y-2) = 0$$

Factorising.

$$\Rightarrow \quad y = -3 \text{ or } y = 2$$

$$\text{So } e^{2x} = -3 \text{ or } e^{2x} = 2.$$

Remember that $y = e^{2x}$.

These two statements can be analysed using logarithmic functions.

$e^{2x} = -3$ has no solutions since $e^{2x} > 0$ for all values of x.

Exponential functions are always positive.

If $e^{2x} = 2$ then $\ln(e^{2x}) = \ln 2$

$$2x \ln e = \ln 2$$

By property 5, $\ln e = 1$.

$$2x = \ln 2$$

$$x = \tfrac{1}{2}\ln 2$$

$$= 0.347 \text{ (3 s.f.)}$$

So the equation $e^{4x} + e^{2x} - 6 = 0$ has only one solution; $x = 0.347$ (3 s.f.).

Check this result on your calculator.

Logarithms can also be used to solve problems involving exponential functions and inequalities.

Example 4

a Find the smallest integer p such that 2.5^p exceeds one million.

b Find the largest integer x such that $(0.7)^x > 5$.

This means solve $2.5^p > 1\,000\,000$.

Solution

a Taking logarithms of both sides,

$$\log_{10} 2.5^p > \log_{10} 1\,000\,000$$

$$\Rightarrow p \log_{10} 2.5 > 6$$

$$\Rightarrow \quad p > \frac{6}{\log_{10} 2.5}$$

$$\Rightarrow \quad p > 15.077$$

Note that $\log_{10}$ is preferable here because 1 000 000 is a power of 10.

Check that $2.5^{16} > 1 \times 10^6$ and $2.5^{15} < 1 \times 10^6$.

Since p has to be an integer, $p = 16$.

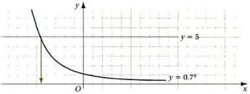
b Taking logarithms of both sides,

$$\log_{10}(0.7^x) > \log_{10} 5$$
$$\Rightarrow \quad x \log_{10} 0.7 > \log_{10} 5$$
$$\Rightarrow \quad x < \frac{\log_{10} 5}{\log_{10} 0.7}$$
$$\Rightarrow \quad x < -4.512$$

Remember to reverse the inequality becaus $\log_{10} 0.7$ is negative.

Since x must be an integer, $x = -5$.

Check that the negative result is sensible. This can be done graphically by considering the intersection of $y = 0.7^x$ and $y = 5$. Check this result on a graphical calculator.



Sometimes x can appear as a power on both sides of an equation. To solve this logarithms can still be used.

Example 5

Find a value of x such that $5^x = 7^{(x-2)}$.

Solution

Start by taking logarithms of both sides:

$$\ln 5^x = \ln 7^{(x-2)}$$
$$\Rightarrow \quad x \ln 5 = x \ln 7 - 2 \ln 7$$

Using property 2.

$$\Rightarrow \quad x \ln 5 - x \ln 7 = -2 \ln 7$$
$$\Rightarrow \quad x(\ln 5 - \ln 7) = -2 \ln 7$$

Collecting like terms and factorising.

$$\Rightarrow \quad x = -\frac{2 \ln 7}{\ln 5 - \ln 7} = \frac{2 \ln 7}{\ln 7 - \ln 5}$$
$$\Rightarrow \quad x = 11.6 \text{ (3 s.f.)}$$

Compare $5^{11.6}$ and 7^9 on your calculator.

Graphical applications

We have looked at techniques of solving problems of the form $a^x = b$. These techniques also have a graphical application. If an exponential relationship exists between two sets of data, then logarithms can be used to simplify their exponential graph into a straight line. This technique is called **reduction to linear form**. There are two basic cases to consider:

- the exponent is variable
- the exponent is constant.

'Exponent', 'index' and 'power' all mean the same.

The exponent is variable

This means that the relationship between x and y is such that $y = a \times b^x = ab^x$ where a and b are constants. By using the logarithm techniques:

$\log y = \log ab^x$

$\log y = \log a + \log b^x$ ◀ **By property 1.**

$\log y = \log a + x \log b$ ◀ **By property 2.**

Look carefully at this result. What do you notice? It has the form $Y = mX + c$ where $Y = \log y$, $X = x$, $m = \log b$ and $c = \log a$. So plotting $\log y$ vertically against x horizontally should give a straight line of gradient $\log b$ with a vertical intercept of $\log a$.

Which is the form of a *linear* equation.

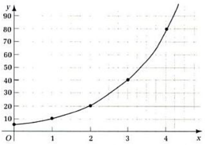
Example 6

The relationship between the values of x and y in the following table is $y = ab^x$, where a and b are positive integers. Find a and b.

x	0	1	2	3	4
y	5	10	20	40	80

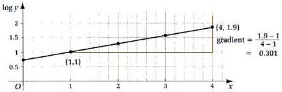
Solution

Plotting the data on a graph gives an exponential style curve.



However, when $\log y$ is plotted against x a straight line is obtained.

x	0	1	2	3	4
$\log y$	0.699	1.000	1.301	1.602	1.903



From the straight-line graph notice that the intercept with the 'y-axis' is approximately 0.7 and the gradient is about 0.301. We have already seen that an exponential equation in the form $y = ab^x$ can be written in the linear form $\log y = \log a + x \log b$, where $\log b$ is the gradient of the line $\log y = \log a + x \log b$, and $\log a$ is the intercept with the vertical axis.
From the graph, $\log a \approx 0.7$ and $\log b \approx 0.301$.
So $a = 5$ and $b = 2$
So the equation that fits the data is $y = 5 \times 2^x$.

Check this model against the original d

The exponent is constant

This means that the relationship between x and y is such that $y = a \times x^b = ax^b$, where a and b are constants. This is really a power function and not an exponential function. Using the logarithm properties,

$$\log y = \log(ax^b)$$

$$\log y = \log a + \log x^b \quad \blacktriangleleft \textbf{ By property 1.}$$

$$\log y = \log a + b \log x \quad \blacktriangleleft \textbf{ By property 2.}$$

What do you notice about this result? This also has the linear form $Y = mX + c$ if $\log y$ is plotted vertically as Y and $\log x$ is plotted horizontally as X. The resulting straight line will have a gradient of b and a vertical intercept of $\log a$.

In the early ninete century, Charles Babbage and fellow Cambridge student J Herschel recalculated corrected logarithm t which at that stage w notoriously inaccurat This was laborious, tedious work and dem a high degree of accu Babbage once made seemed to be a passir comment, 'It is a pity can't be done by stean later went on to develo number of machines, including the first calcu which he called his 'analytical engine'.

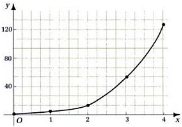
Example 7

The relationship between x and y in the following table of values is $y = ax^b$ where a and b are positive integers. Use a graphical method to find a and b.

x	0	1	2	3	4
y	0	2	16	54	128

Solution

The data plotted on a graph gives a curve.

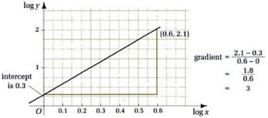


Now take logarithms and retabulate the data.

Note that the point $x = 0, y = 0$ cannot be used.

$\log x$	0	0.301	0.477	0.602
$\log y$	0.301	1.204	1.732	2.107

Plotting $\log y$ against $\log x$ gives a straight-line graph.



From the graph notice that the vertical intercept is 0.3 and the gradient is 3. From the linear equation in $\log x$ and $\log y$, $\log y = \log a + b \log x$, this means $\log a \approx 0.3$ and $b \approx 3$.

So $a = 2$ and $b = 3$.

Thus the equation that fits this data is $y = 2x^3$.

Remember that a and b are integers.

When you get sets of data like these a decision has to be made. Which model should be used? If either of the graphs gives a straight line then an exponential or power form would be the expected equation. One way of saving time is to look at the original data and determine if it goes through the origin. If it does, then suspect that the exponent is constant.

Check this model against the original data.

7.4 Solving $a^x = b$

Exercise

Technique

1 Solve the following equations:

a $2^x = 4000$ **b** $5^x = 500$
c $11^x = 151$ **d** $4^x = 17$

2 Solve the equations:

a $3^{x+1} = 25$ **b** $4^{x-3} = 30$
c $5^{2x+3} = 51$ **d** $3^{2x-1} = 29$

3 Find the value of x in the following:

a $5^{3x} = 7^{x-2}$ **b** $2^{x+1} = 3^x$
c $7^{x+3} = 3^{x-2}$ **d** $2^{3x+1} = 3^{2x-3}$

3 a

4 Solve the following exponential equations:

a $e^{2x} - 5e^x + 6 = 0$ **b** $2e^{2x} - 9e^x + 4 = 0$
c $e^{2x} + e^x - 6 = 0$ **d** $6e^{4x} - 13e^{2x} - 5 = 0$

5 Find the smallest integer p such that:

a $2^p > 1000$ **b** $3^p > 2000$
c $(0.9)^p < 0.0001$ **d** $(0.5)^p < 0.001$

6 Find equations for the following sets of data:

a

x	0	1	2	3	4
y	3	15	75	375	1875

b

x	0	1	2	3	4
y	0	3	96	729	3072

c

x	0	0.5	1.0	1.5	2.0
y	2	3.46	6	10.4	18

d

x	0	0.5	1.0	1.5	2.0
y	3	1.10	0.406	0.149	0.0549

Hint: Use ln in part **d**.

7 Solve the following equations:

Hint: Let $y = 10^x$.

a $100^x - 10^{x+1} + 16 = 0$ **b** $10^{2x} - 10^{x+1} + 21 = 0$
c $10^{2x} - 10.10^x + 24 = 0$ **d** $10^{2x} - 100 = 0$

7 a

Contextual

1 The population, P thousands, of a new town is calculated every t years after 1979. The results are summarised in the table below.

t	1	2	3	4	5
P	12.1	18.4	28.0	42.5	64.6

It is believed that P and t are connected by an exponential relationship of the form $P = ab^t$, where a and b are constants. Verify this graphically and find the values of a and b. When is the population expected to exceed half a million?

2 Show that the equation $e^{2x} - 7e^x - 8 = 0$ has one real solution. Why does the second solution of the related quadratic not produce a solution to the exponential equation?

3 Before Newton's theory of gravitation, the best mathematical model to describe planetary motion was formulated by Kepler (1571–1630). Kepler stated three laws, the third of which gave a relationship between the distance of the planet from the sun, R (millions of km), and the period of its orbit, T (years). Kepler had data for the following six planets.

Planet	Distance from the Sun (millions of km)	Period of orbit (years)
Mercury	57.9	0.241
Venus	108.2	0.616
Earth	149.6	1.0
Mars	227.9	1.881
Jupiter	778.3	11.852
Saturn	1427.0	29.440

Using the model $T = aR^b$, where a and b are constants, find Kepler's third law. Check this model on the following data for planets not known to Kepler:

Planet	Distance (R)	Period of orbit (T)
Uranus	2870	83.943
Neptune	4497	164.681
Pluto	5907	248.241

How accurate is the model?

4 Find the real value of k for which $10^x = e^{kx}$ for all values of x.

Consolidation

Exercise A

1 Without using a calculator show that $\dfrac{\log 9\sqrt{3} - \log 4\sqrt{2}}{\log 3 - \log 2} = \dfrac{5}{2}$.

2 Solve the following inequalities:

a $\ln(x + 2) < 3$ **b** $(0.7)^x > 3$

3 The value of a car (in £) can be modelled by the equation $V = 8500e^{-\lambda t}$, where t is the age of the car in years and λ is a constant.

a State the value of the car when it was new.
b After two years the value of the car was £6580. Use this information to calculate the value of λ.
c Estimate the value of the car after three years.
d How long would it take for the value of the car to be half its original value?

4 **a** If $A_0 = 100$ and $A_{n+1} = 1.07A_n$ find a formula for A_n in terms of n.
b If n represents the number of years a sum of money is left in an account and A_0 represents an initial investment of £100, what does 1.07 represent?
c How many years would it take for the investment to double?

5 Given that $y = 10^x$, show that:

a $y^2 = 100^x$
b $\frac{y}{10} = 10^{x-1}$
c Using the results from **a** and **b** write the equation $100^x - 10\,001(10^{x-1}) + 100 = 0$ as an equation in y.
d By first solving the equation in y, find the values of x which satisfy the given equation in x.

(ULEAC)

6 The population of Portugal t years after 1990 can be modelled by the equation $P = P_0e^{kt}$.

a In 1990 the population was estimated to be 10.5 million. Write down the value of P_0.
b The growth rate was expected to be 0.5% per annum. Explain why the population projection for 1991 is $1.05 \times 10^7 \times (1.005)$.
c Use the result from **b** to find the value of k correct to three significant figures.
d Use the result from **c** to predict the population for 2000.

7 Given that $y = \log_b 45 + \log_b 25 - 2\log_b 75$, express y as a single logarithm in base b. In the case when $b = 5$, state the value of y.

(AEB)

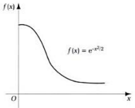
8 For certain planets, the approximate mean distance x, in millions of kilometres from the centre of the Sun, and the orbit T, in Earth years, are recorded.

x	57.9	108.2	227.9	778.3
T	0.24	0.62	1.88	11.86

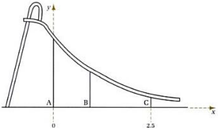
Assuming a law of the form $T = Ax^n$, draw a graph of $\ln T$ against $\ln x$. Estimate the values of A and n, giving your answers to two significant figures. Use your graph to estimate the approximate mean distance in millions of kilometres of the Earth from the Sun.

(AEB)

9 The graph shows the function $f(x) = e^{-x^2/2}$ for $x > 0$. By writing $y = e^{-x^2/2}$ find an expression for x in terms of y. Hence write down $f^{-1}(x)$.



10 A slide into a ball pool at a children's play centre can be modelled by the curve $y = 3 \times 10^{-x/2}$ where x and y are measured in metres. The slide has three supports, fixed to the ground at positions A, B and C as shown in the diagram.



a Calculate the height of the supports at A and C.

b The support at B is half the height of the support at A. How far along the ground is support B from support A.

Exercise B

1 Without using a calculator express as a single logarithm,

$$2\log\left(\frac{5}{\sqrt{2}}\right) + \log 3 + 2\log 2$$

2 Solve the following inequalities:

a $\ln(2x) < 5$ **b** $(0.95)^{x+1} > 8$

3 The value V of a particular car, in pounds, at age t months from new can be modelled by the equation $V = 12000e^{-kt} + 2000$, where k is a constant.

a Use this model to write down the value of the car when new.

b The value of the car is expected to be £8000 after 24 months. On this basis, calculate the value of k.

(AEB)

4 An employee joins a large firm on 1 January of a particular year. In order to provide a sum of money for the employee's retirement, £1500 is paid into a special account on 1 January each year for the previous year's work. In addition, on each 1 January, interest is added to the account, the interest being 6% of the total amount in the account immediately prior to the annual payment of £1500.

a Show that for an employee, who retires on 1 January after serving n years with the firm, the sum of money in the special account is equal to £25 000$(1.06^n - 1)$.

b Find the least number of years required for the sum to exceed £100 000.

(WJEC)

5 By treating the following equation as a quadratic in e^x, find the two values of x satisfying $e^{2x} - 5e^x + 6 = 0$.

(WJEC)

6 The population of Iceland t years after 1990 can be modelled by the equation $P = 0.251e^{\lambda t}$, where P is measured in millions.

a If the growth rate is estimated to be 0.8% per annum, estimate the value of λ.

b In what year would the population reach half a million?

7 Given than $\ln(3x - 5) - \ln 4 = 2\ln y$:

a find the value of y when $x = 2$

b express x in terms of y in a form not involving logarithms.

(AEB)

Applications and Activities

1 *Fibonacci numbers*

This sequence of numbers owes its name to the Italian mathematician Leonardo da Pisa (1170–1230). Each term of the sequence is the sum of the two previous terms. Since $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$ and so on the sequence becomes 1, 1, 2, 3, 5, 8, 13, ...
If these numbers are tabulated,

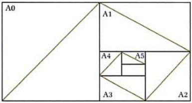
x	1	2	3	4	5	6	7
y	1	1	2	3	5	8	13

and then plotted graphically, a curve is formed.
Find the equation for the curve generated by the Fibonacci numbers.

Hint: Try $y = ab^x$.

2 *The logarithmic spiral*

This spiral intersects all radii at a fixed angle and the distances from its pole increase in a geometric sequence. To sketch a logarithmic spiral quickly use European DIN paper size A0. Fold the A0 size sheet of paper in descending 'A' size order and draw a line from one corner to the opposite corner of each section in a clockwise or anticlockwise order.



Logarithmic spirals occur often in nature. Investigate!

Descartes (1596–1650)
The logarithmic spiral was first recognised by Descartes and found so fascinating by Jacob Bernoulli (1654–1705) that he arranged to have it engraved on his tombstone.

Summary

- If $y = a^x$ then $x = \log_a y$.
- **Common logarithms** have base 10, and are written as $\log_{10}$ or log.
- **Natural logarithms** have base e, and are written as $\log_e$ or ln.
- The five basic properties of logarithms for all bases are:

$$\log ab = \log a + \log b$$

$$\log a^n = n \log a$$

$$\log \frac{a}{b} = \log a - \log b$$

$$\log 1 = 0$$

$$\log_a a = 1$$

- e^x is the **exponential function** and is used to model growth and decay.
- Relationships of the form $y = ab^x$ can be transformed from an exponential curve to a straight line by plotting $\log y$ vertically against x horizontally. The gradient of the straight line is $\log b$ and the y-intercept is $\log a$.
- Relationships of the form $y = ax^b$ can be transformed from an exponential curve to a straight line by plotting $\log y$ vertically against $\log x$ horizontally. The gradient of the straight line is b and the y-intercept is $\log a$.

8 Sequences and Series

What you need to know

- How to collect 'like terms'.
- How to expand brackets.
- How to substitute values into expressions.
- How to express an algebraic fraction in terms of its partial fractions.

Review

1 By collecting together like terms, simplify the following expressions:

a $a + a + 2d$ **b** $a + a + (n-1)d$
c $a + ar + ar$ **d** $1 + x^2 + x^3 - \frac{1}{2}(x^2 + x^3)$

2 Expand the following:

a $(n+3)(n+4)$ **b** $(r+2)(r-1)$
c $r(r+1)$ **d** $(1+x)^2$

3 Evaluate:

a $2n + 3$, when $n = 3$
b $3n - 1$, when $n = 5$
c $n^2 - 1$, when $n = 4$
d $\frac{1}{2}n(n-1)(n-2)$, when $n = 3$
e ar^4, when $a = \frac{1}{2}$, $r = 2$
f $\dfrac{a(1-r^n)}{(1-r)}$, when $a = 10$, $r = \frac{1}{2}$, $n = 4$.

4 Express each of the following in terms of their partial fractions:

a $\dfrac{5}{r(r+1)}$ **b** $\dfrac{6}{(r+1)(r+2)}$
c $\dfrac{7}{(n+1)(n+2)}$ **d** $\dfrac{12}{n(n+1)}$

8.1 Sequences and Series

Sequences

A **sequence** is a set of numbers occurring in order. Some examples of sequences are:

$2, 4, 6, 8, \ldots$	the sequence of even numbers
$1, 4, 9, 16, \ldots$	the sequence of perfect squares
$1, 3, 6, 10, \ldots$	the sequence of triangular numbers.

The terms of a sequence are often represented by ordered lower-case letters. Thus u_1 denotes the first term and u_2 denotes the second, and the sequence becomes:

$$u_1, u_2, u_3, \ldots$$

For the sequence of even numbers, $u_1 = 2$, $u_2 = 4$, $u_3 = 6$, and so on.
A sequence can be described in several different ways. One method is to give an algebraic expression for **the *n*th term**. The nth term for the sequence of even numbers is $2n$. This can be written:

$$u_n = 2n$$

Check that $u_n = n^2$ and $u_n = \frac{1}{2}n(n+1)$ generate the sequence of perfect squares and the sequence of triangular numbers respectively.

Example 1

Find the first five terms of the sequence whose nth term is given by

$$u_n = \frac{n(n-1)(n-2)}{6}$$

Solution

To find the first five terms put $n = 1, 2, 3, 4$ and 5 respectively into the expression for u_n.

$$u_1 = \frac{1(1-1)(1-2)}{6} = 0$$

$$u_2 = \frac{2(2-1)(2-2)}{6} = 0$$

$$u_3 = \frac{3(3-1)(3-2)}{6} = 1$$

$$u_4 = \frac{4(4-1)(4-2)}{6} = 4$$

$$u_5 = \frac{5(5-1)(5-2)}{6} = 10$$

So the first five terms are 0, 0, 1, 4, 10.

There are other types of sequences where the sign oscillates between positive and negative.

Example 2

Find the nth term of the sequence $-1, 3, -5, 7 \ldots$

Solution

Notice that without the change of sign this sequence is the sequence of odd numbers. For odd numbers, $u_n = 2n - 1$.
The change of sign in the sequence can be generated by using powers of -1.

$(-1)^1 = -1$ ◀ **Odd powers give negative results.**

$(-1)^2 = +1$ ◀ **Even powers give positive results.**

$(-1)^3 = -1$

$(-1)^4 = +1$

So the nth term of the sequence $-1, 3, -5, 7$ is given by

$$u_n = (-1)^n(2n - 1)$$

Verify this result for $n = 1, 2, 3$ and 4.

Sometimes the rule for defining a sequence can be given in the form of a **recurrence relation**. This means the nth term u_n can be calculated only when the preceding terms are known.

Example 3

The **Fibonacci sequence**, $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ can be defined by a recurrence relation. Find this rule.

The Fibonacci sequence was discovered in the early thirteenth century by Leonardo of Pisa who used the name Fibonacci.

Solution

To find the recurrence relation look at how the terms of the sequence are related to each other.

$u_1 = 1$

$u_2 = 1$

$u_3 = 2 = u_2 + u_1$

$u_4 = 3 = u_3 + u_2$

$u_5 = 5 = u_4 + u_3$

$u_6 = 8 = u_5 + u_4$

So $u_n = u_{n-1} + u_{n-2}$

The nth term in the Fibonacci sequence can only be found if the previous two terms, u_{n-1} and u_{n-2}, are known.

Example 4

a Find the first four terms of the sequence defined by the recurrence relation $u_n = u_{n-1} + 2$ with $u_1 = 3$.

b Find the first five terms of the sequence defined by $u_n = 3u_{n-1} - u_{n-2}$, with $u_1 = 1$ and $u_2 = 2$.

Solution

a Given u_1, find u_2 using the given recurrence relation. Then calculate u_3 and u_4 using the same technique.

$u_1 = 3$

$u_2 = u_1 + 2 = 5$

$u_3 = u_2 + 2 = 7$

$u_4 = u_3 + 2 = 9$

So the first four terms of the sequence are 3, 5, 7, 9.

b Use the values of u_1 and u_2 to find u_3. Then generate u_4 and u_5 using the same recurrence relation.

$u_1 = 1$

$u_2 = 2$

$u_3 = 3u_2 - u_1 = 6 - 1 = 5$

Remember to put $n =$ in the recurrence relation.

$u_4 = 3u_3 - u_2 = 15 - 2 = 13$

$u_5 = 3u_4 - u_3 = 39 - 5 = 34$

So the first five terms of the sequence are 1, 2, 5, 13, 34.

Series

When the terms of a sequence are added together, a **series** is formed.

$2 + 4 + 6 + 8$	is a finite series
$1 + 4 + 9 + 16 + \cdots$	is an infinite series

If a sequence has five terms, u_1, u_2, u_3, u_4 and u_5, then the series based on this sequence would be

$$u_1 + u_2 + u_3 + u_4 + u_5$$

This series can be written in a more concise form. Instead of appearing as a string of terms added together, the series is written using **sigma notation**.

$$u_1 + u_2 + u_3 + u_4 + u_5 = \sum_{r=1}^{5} u_r$$

Σ is the Greek letter, sigma. It is used in mathematics to represent a summation. The letter r is a **dummy variable**. The $r = 1$ at the foot of the sigma indicates the term with which the summation should start, and the 5 at the top indicates the term with which the summation should finish.

Thus $\sum_{r=4}^{7} u_r$ means start with the fourth term, u_4 and sum to the seventh term, u_7.

Example 5

If $u_r = 2r - 1$, find $\sum_{r=1}^{5} u_r$.

Solution

To sum this series, first substitute the values of r from 1 to 5 into the expression for the rth term of the sequence and then sum the results.

$$\begin{aligned}\sum_{r=1}^{5} u_r &= \sum_{r=1}^{5} (2r - 1)\\ &= (2-1) + (4-1) + (6-1) + (8-1) + (10-1)\\ &= 1 + 3 + 5 + 7 + 9\\ &= 25\end{aligned}$$

Sigma notation is particularly useful when the series is infinite. In these cases there is no last term, so writing the series as a summation string would be impossible. In sigma notation, however, we can write:

$$\sum_{r=1}^{\infty} u_r = u_1 + u_2 + u_3 + \cdots$$

So the sum of the even numbers can be written:

$$2 + 4 + 6 + 8 + \cdots = \sum_{r=1}^{\infty} 2r$$

Writing a series in sigma notation is dependant upon an algebraic expression for the nth term being known.

Example 6

Express:

a $1 + 3 + 6 + 10 + 15$ in sigma notation

b $7 + 10 + 13 + 16$ in the form $\sum_{r=3}^{n} f(r)$

Solution

a The terms in this series are from the sequence of triangular numbers. The nth term, $u_n = \frac{1}{2}n(n+1)$. Since $1 + 3 + 6 + 10 + 15$ begins with the first term and ends with the fifth term,

$$1 + 3 + 6 + 10 + 15 = \sum_{r=1}^{5} \frac{r(r+1)}{2}$$

Recall that r is a dum
variable.

b The first step is to find an expression for the nth term. Notice that the sequence 7, 10, 13, 16 increases in equal steps of 3. This suggests the expression for the nth term should contain a 3. Rewrite the sequence using this factor of 3. Note that $r = 3$ for the first term.

$$u_3 = 7 \qquad u_4 = 10 \qquad u_5 = 13$$
$$= 3 \times 3 - 2 \qquad = 4 \times 3 - 2 \qquad = 5 \times 3 - 2$$

This suggests that we can write the nth term as

$$u_n = n \times 3 - 2 = 3n - 2$$

The series is finite with last term 16.

Hence $\quad 3n - 2 = 16$

$$3n = 18$$
$$n = 6$$

Therefore $\quad 7 + 10 + 13 + 16 = \sum_{r=3}^{6} (3r - 2)$

8.1 Sequences and Series Exercise

Technique

1 Find the first four terms of the sequences with the following nth terms:

a $u_n = 3n + 4$ **b** $u_n = 4n - 1$ **c** $u_n = 6n - 1$
d $u_n = (n+3)(n+4)$ **e** $u_n = 2^{n-1}$ **f** $u_n = n^3 + 1$

2 Find u_n, the nth term, for the following sequences:

a $6, 8, 10, 12, \dots$ **d** $12, 10, 8, 6, \dots$ **g** $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \dots$
b $14, 17, 20, 23, \dots$ **e** $-3, -5, -7, -9, \dots$ **h** $0, \frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \dots$
c $1, 6, 11, 16, 21, \dots$ **f** $0, 3, 8, 15, 24, \dots$

3 Use the following recurrence relations to find the next four terms of each sequence:

a $u_n = u_{n-1} + 3, u_1 = 1$ **d** $u_n = 2u_{n-1} - u_{n-2}, u_2 = 2, u_1 = 1$
b $u_n = 2u_{n-1} + 1, u_1 = 1$ **e** $u_n = 2u_{n-1} + u_{n-2}, u_2 = 2, u_1 = 1$
c $u_n = 3u_{n-1} - 1, u_1 = 2$ **f** $u_n = \frac{1}{2}(u_{n-1}) + 1, u_1 = 4$

3 c

4 Express the following series in sigma notation:

a $1 + 4 + 9 + 16$ **d** $3 + 5 + 7 + 9 + 11 + 13$
b $3 + 6 + 11 + 18$ **e** $0 + 2 + 6 + 12 + 20 + 30 + 42$
c $4 + 5 + 6 + 7 + 8 + 9$ **f** $2 + 6 + 12 + 20 + 30 + 42 + 56$

5 Evaluate:

a $\sum_{r=1}^{3} r^3$ **b** $\sum_{r=2}^{4} (r^2 + 1)$ **c** $\sum_{r=3}^{5} r^3$

d $\sum_{r=4}^{6} (5r - 2)$ **e** $\sum_{r=3}^{6} r(r+1)$ **f** $\sum_{r=3}^{5} (r+1)(r+2)$

6 Express the following series in the given form of sigma notation:

a $5 + 8 + 11 + 14, \quad \sum_{r=3}^{n} f(r)$

b $5 + 8 + 11 + 14 + 17 + 20, \quad \sum_{r=3}^{n} f(r)$

c $9 + 11 + 13 + 15, \quad \sum_{r=2}^{n} f(r)$

d $9 + 15 + 21 + 27, \quad \sum_{r=4}^{n} f(r)$

e $7 + 10 + 13 + 16 + 19, \quad \sum_{r=2}^{n} f(r)$

f $6 + 8 + 10 + 12 + 14 + 16 + 18, \quad \sum_{r=3}^{n} f(r)$

8.2 Arithmetic Progression

An **arithmetic progression** is a sequence in which each term is produced by adding a fixed number. For example $2, 5, 8, 11, \ldots$ is an arithmetic progression and $10, 7, 4, 1, -2, \ldots$ is also an arithmetic progression. Notice that the number that is added throughout the sequence can be positive or negative. It is called the **common difference**.

The phrase 'arithmetic progression' is often abbreviated to AP. The first term is usually denoted by a and the common difference by d.

Consider the AP $2, 5, 8, 11, \ldots$. Here $a = 2$ and $d = 3$. This sequence can now be rewritten as

$$2, \quad 2+3, \quad 2+(2\times 3), \quad 2+(3\times 3), \quad 2+(4\times 3), \ldots$$

The nth term would be $2 + (n-1) \times 3$.

In general the nth term of an AP is $u_n = a + (n-1)d$

◀ **Learn this important result.**

Example 1

Find an expression for the nth term of the AP $10, 7, 4, 1, -2, \ldots$.

Solution

For this AP, $a = 10$ and $d = -3$.

$$\begin{aligned} u_n &= a + (n-1)d \\ &= 10 + (n-1) \times (-3) \\ &= 10 - 3n + 3 \\ &= 13 - 3n \end{aligned}$$

Check this expressior for u_n for values of n from 1 to 5.

The terms of an AP can be added to form an **arithmetic series**.

$$a + (a+d) + (a+2d) + (a+3d) + \cdots + (a+(n-1)d) + \cdots$$

If this series is finite then it has a last term and it is possible to find an expression for the sum of the series.

Example 2

Find the sum of the first 100 natural numbers.

One technique f finding this su first demonstrated b Gauss (1777–1855 age nine he mentally calculated the total a 5050.

Solution

The first 100 natural numbers form an AP, with $a = 1$ and $d = 1$. The sum of this AP is a finite arithmetic series:

$$1 + 2 + 3 + 4 + \cdots + 100$$

How can this total be calculated?

Let S_{100} be the sum of the first 100 terms.

$$S_{100} = 1 + 2 + 3 + \cdots + 99 + 100 \qquad [1]$$

Now write the sum again in reverse.

$$S_{100} = 100 + 99 + 98 + \cdots + 2 + 1 \qquad [2]$$

Adding [1] and [2], term by term,

$$2 \times S_{100} = 101 + 101 + 101 + \cdots + 101 + 101$$

We know that there are 100 terms on the RHS, so

$$2 \times S_{100} = 100 \times 101$$

$$\Rightarrow \quad S_{100} = \tfrac{1}{2}(100 \times 101) = 5050$$

The same idea can be applied to every finite arithmetic series.

$$S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d)$$

$$S_n = [a + (n-1)d] + \cdots + (a+2d) + (a+d) + a$$

$$2S_n = [2a + (n-1)d] + \cdots + [2a + (n-1)d]$$

There are n terms on the RHS, so

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

◀ **Learn this important result.**

Graphical calculator support pack

The sum to n terms of an arithmetic series can be written in another equivalent form:

$$S_n = \frac{n(a+l)}{2}$$

a = first term
l = last term

Example 3

For the AP $5, 9, 13, 17, \ldots$, find:

a the sixth term, u_6
b the sum of the first five terms, S_5.

Solution

First identify the first term and the common difference. The standard results for nth term and the sum to n terms can then be applied.
For the AP $5, 9, 13, 17, \ldots$, $a = 5$ and $d = 4$

a $u_n = a + (n-1)d$

$$\Rightarrow u_6 = 5 + (6-1) \times 4 = 5 + (5 \times 4) = 25$$

b $S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_5 = \frac{5}{2} \times [(2 \times 5) + (5-1) \times 4] = \frac{5}{2} \times (10 + 16) = 65$$

Example 4

Find the sums of the following arithmetic series:

a $2 + 5 + 8 + 11 + \cdots + 47$ **b** $47 + 41 + 35 + \cdots + (-43)$

Solution

a Here $a = 2$ and $d = 3$.
The number of terms in the series, n, can be found by making the last term, 47, the nth term.

$$u_n = a + (n-1)d$$
$$\Rightarrow 47 = 2 + (n-1) \times 3 = 3n - 1$$
$$\text{So } 3n = 47 + 1 \Rightarrow n = 16$$

Remember to check t the sequences are AP Identify the first term and common differences *before* usi any other formula.

The sum to 16 terms of the series can be found by using one of the formulas for S_{16}.

$$S_{16} = \frac{n(a+l)}{2} = \frac{16(2+47)}{2} = 392$$

b Here $a = 47$ and $d = -6$. Making -43 the nth term,

$$u_n = a + (n-1)d$$
$$\Rightarrow -43 = 47 + (n-1)(-6)$$
$$\Rightarrow -43 = 47 - 6n + 6 = 53 - 6n$$
$$\text{So } 6n = 53 + 43 \Rightarrow n = 16$$

This arithmetic series also has 16 terms.

$$S_{16} = \frac{n(a+l)}{2} = \frac{16[47 + (-43)]}{2} = 32$$

Why is this total so low? Recall that many of the terms were negative.

$$S_{16} = 47 + 41 + 35 + \cdots - 25 - 31 - 37 - 43$$

This technique of identifying the first term, a, and common difference, d, for the AP is particularly useful when the information given relates to other terms in the series.

Example 5

The fifth term of an AP is twice the second term. The two terms differ by 9. Find the sum of the first 10 terms of the AP.

Solution

The information given is that

$$u_5 = 2u_2 \quad \text{and} \quad u_5 - u_2 = 9$$

Using the fact that the nth term of an AP is given by $u_n = a + (n-1)d$, these two equations can be rewritten as simultaneous equations in a and d.

$$\begin{aligned}
u_5 = 2u_2 &\Rightarrow & u_5 - 2u_2 &= 0 \\
&\Rightarrow & (a+4d) - 2(a+d) &= 0 \\
&\Rightarrow & a + 4d - 2a - 2d &= 0 \\
&\Rightarrow & 2d - a &= 0 & [1] \\
u_5 - u_2 = 9 &\Rightarrow & (a+4d) - (a+d) &= 9 \\
&\Rightarrow & a + 4d - a - d &= 9 \\
&\Rightarrow & 3d &= 9 \\
&\Rightarrow & d &= 3 & [2]
\end{aligned}$$

Notice that equation [2] gives the value of d. Substituting this value of d into equation [1] gives

$$(2 \times 3) - a = 0 \Rightarrow a = 6$$

The sum of the first 10 terms can now be found. Use these values of a and d with $n = 10$ in the formula for the sum to n terms.

$$\begin{aligned}
S_n &= \tfrac{1}{2}n(2a + (n-1)d) \\
\text{So } S_{10} &= \tfrac{10}{2}(12 + (9 \times 3)) \\
&= 5(12 + 27) = 195
\end{aligned}$$

Example 6

Evaluate $\sum_{r=3}^{16}(2r+1)$.

Recall how the sigma notation works; r is a dummy variable.

Solution

$$\begin{aligned}
\sum_{r=3}^{16}(2r+1) &= (6+1) + (8+1) + \cdots + (32+1) \\
&= 7 + 9 + \cdots + 33
\end{aligned}$$

This arithmetic series is generated by an AP where $a = 7$ and $d = 2$. Check that there are 14 terms in the series. The sum of the series can now be calculated.

Hint: Reduce $r = 3$ to $r = 1$ in the dummy variable.

$$S_n = \tfrac{1}{2}n[2a + (n-1)d]$$

So $$S_{14} = \tfrac{14}{2}[(2 \times 7) + (13 \times 2)]$$

$$= 7(14 + 26) = 280$$

So $\sum_{r=3}^{16}(2r+1) = 280$

Example 7

Pat saves £10 in the first month, £12 in the second month and increases the monthly savings by £2 each month. How long will it take Pat to save £500?

Solution

Notice that the monthly savings form an AP in which $a = 10$ and $d = 2$. Suppose Pat saves £500 in n months. This means the sum of the first n terms of the AP is 500.

So $$S_n = 500$$

$\Rightarrow \tfrac{1}{2}n(20 + 2(n-1)) = 500$ ◀ **Recall that $a = 10$ and $d = 2$.**

$\Rightarrow n(20 + 2n - 2) = 1000$

$\Rightarrow 18n + 2n^2 = 1000$

$\Rightarrow 2n^2 + 18n - 1000 = 0$

This is a quadratic equation in n. Solve it using the quadratic formula.

$n = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$n = \frac{-18 \pm \sqrt{18^2 - 4 \times 2 \times (-1000)}}{2 \times 2}$$

$$= 18.3 \quad \text{or} \quad -27.3 \text{ (3 s.f.)}$$

The negative answer can be ignored because the number of months cannot be negative.
So £500 is saved in 18.3 months. Given that Pat saves monthly, the savings exceed £500 after 19 months.

The number of mont
must be an integer.

8.2 Arithmetic Progression
Exercise
Technique

1 Find **(i)** the next two terms and **(ii)** the nth term u_n, of each of the following arithmetic progressions:

a $13, 15, 17, 19, \ldots$ **b** $8, 14, 20, 26, \ldots$
c $34, 31, 28, 25, \ldots$ **d** $-2, -5, -8, -11, \ldots$

2 Given the arithmetic progressions $3, 7, 11, 15, \ldots, u_n$; $6, 11, 16, 21, \ldots, v_n$; and $70, 66, 62, 58, \ldots, w_n$, find:

a u_6 **b** u_{30} **c** v_{11}
d v_{100} **e** w_9 **f** w_{30}

3 Find the sums of the following arithmetic series:

a $1 + 5 + 9 + 13 + \cdots + 41$ **b** $2 + 8 + 14 + 20 + \cdots + 38$
c $4 + 7 + 10 + 13 + \cdots + 25$ **d** $22 + 27 + 32 + \cdots + 47$

4 Use the method of Gauss (Example 2) to calculate:

a $16 + 14 + 12 + \ldots + 2$ **d** $28 + 27 + 26 + \ldots + 17$
b $27 + 24 + 21 + \ldots + 9$ **e** $22 + 17 + 12 + \ldots - 18$
c $56 + 52 + 48 + \ldots + 32$ **f** $-3 - 5 - 7 - 9 \ldots - 21$

5 Evaluate:

a $\sum_{r=1}^{10} (3r + 2)$ **b** $\sum_{r=2}^{14} (2r - 1)$
c $\sum_{r=5}^{20} (2r + 5)$ **d** $\sum_{r=6}^{16} (3r - 10)$

5 c

Contextual

1 The number of beams in a bridge structure form a sequence $3, 5, 7, 9, \ldots$. What is the nth term of this sequence?

2 Calculate the 20th term of the sequence $-8, -2, +4, +10, \ldots$.

3 Find the sum of the arithmetic series $15 + 18 + 21 + \cdots + 60$. Check the result using the method of Gauss (Example 2).

4 Find the sum of the first 16 terms of the arithmetic progression $11 + 18 + 25 + 32 + \ldots$.

5 The ninth term of an arithmetic series is three times the second term and the difference between the sixth term and twice the first term is 10. Find:

- **a** the first term
- **b** the common difference of the AP.

6 The sum of the first five terms of an AP is 72. The sum of the first ten terms is 189. Find the sum of the first fifteen terms. 6

7 When x is added to 6, 12, and 14 respectively and the results are squared, the new sequence is an arithmetic progression. Find x.

8 Luke's piggy bank deposits increase by 5p each week. His initial deposit is 10p. How long will it be before Luke's savings are at least:

a £10 **b** £100?

In the week his savings exceeded £100, how much did Luke deposit?

9 An AP has three consecutive terms with a sum of 33 and a product of 935. Find the terms.

10 An AP has first term 10 and common difference 0.8. If the sum of the first n terms is to exceed 250, find the value of n. 10

8.3 Geometric Progression

A **geometric progression** is a sequence in which each term is produced from the preceding term by multiplying by a fixed number. For example $3, 6, 12, 24, \ldots$ is a geometric progression, and so is $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}$. Notice that the multiplier throughout the sequence can be positive or negative. It is known as the **common ratio**.

In general the phrase 'geometric progression' is abbreviated to GP. The first term is denoted by a and the common ratio by r.

A GP can then be written as

$$a, ar, ar^2, ar^3, ar^4, \ldots$$

The common ratio can be found by comparing successive terms in the sequence.

$$\frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \frac{u_5}{u_4} = r$$

In general, the nth term of a GP is $u_n = ar^{n-1}$

◀ **Learn this important result.**

Example 1

Find an expression for the nth term of the GP $3, 6, 12, 24, \ldots$.

Solution

Look closely at the sequence.

$$u_1 = 3,\ u_2 = 6,\ u_3 = 12,\ u_4 = 24$$

So $\frac{u_2}{u_1} = 2$, $\frac{u_3}{u_2} = 2$, and $\frac{u_4}{u_3} = 2$.

This GP has first term 3 and common ratio 2.

$$u_1 = 3,\ u_2 = 3 \times 2,\ u_3 = 3 \times 2^2,\ u_4 = 3 \times 2^3,\ \ldots,\ u_n = 3 \times 2^{n-1}$$

So the nth term is $3 \times 2^{n-1}$

Example 2

Find an expression for the nth term of the GP $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$.

Solution

Notice that for this GP, $a = 1$ and $r = -\frac{1}{2}$.

$$\begin{aligned} u_n &= ar^{n-1} \\ &= 1 \times (-\tfrac{1}{2})^{n-1} \\ &= (-\tfrac{1}{2})^{n-1} \end{aligned}$$

Check this result by substituting values $n = 1, 2, 3, 4$ and 5

The terms in a GP can be added to form a **geometric series**. For example, the GP 3, 6, 12, 24 could be used to create the finite geometric series $3+6+12+24$. It is also possible to find a rule to sum a geometric series. The first term, a, and the common ratio, r, of the GP need to be known.
Let $S = 3+6+12+24$
Since this GP has $a = 3$ and $r = 2$ this can be written as

$$S = 3 + (3\times 2) + (3\times 2^2) + (3\times 2^3) \qquad [1]$$

Notice the use here o $u_n = ar^{n-1}$.

Now multiply each term by the common ratio

$$2S = (3\times 2) + (3\times 2^2) + (3\times 2^3) + (3\times 2^4) \qquad [2]$$

This eliminates man the terms on the RHS

Subtracting equation [1] from equation [2].

$$2S - S = (3\times 2^4) - 3$$

$$S(2-1) = 3(2^4 - 1)$$

$$S = \frac{3(2^4-1)}{(2-1)} = 45$$

Check this result against the sum of the four terms of the geometric series.
This procedure can be generalised to find the sum of any *finite* geometric series.

$$\text{Let } S_n = a + ar + ar^2 + \cdots + ar^{n-1} \qquad [1]$$

$$\text{Then } rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \qquad [2]$$

Subtracting [2] from [1],

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

◀ **Learn this important result.**

Example 3

For the sequence $3, 6, 12, 24, \ldots$, find:

a the next two terms
b the nth term
c the sum of the first five terms.

Solution

The sequence 3, 6, 12, 24 is a GP with $a = 3$ and $r = 2$.

a The next two terms are u_5 and u_6. Using $u_n = ar^{n-1}$,

$$u_5 = 3 \times 2^4 = 3 \times 16 = 48$$

$$u_6 = 3 \times 2^5 = 3 \times 32 = 96$$

b The nth term is u_n.

$$u_n = ar^{n-1} = 3 \times 2^{n-1}$$

c The sum of the first five terms is S_5.

$$\begin{aligned} S_5 &= \frac{a(1-r^5)}{1-r} \\ &= \frac{3(1-2^5)}{1-2} \\ &= \frac{3(1-32)}{-1} \\ &= \frac{3(-31)}{-1} \\ &= 93 \end{aligned}$$

Notice that in Example 3c the formula for S_5 created a negative numerator and denominator. This was because the common ratio, r, was greater than 1. In order to reduce problems with negative results an alternative form of the sum to n terms can be used.

If $r > 1$, then use $S_n = \dfrac{a(r^n - 1)}{r - 1}$

If $r < 1$, then use $S_n = \dfrac{a(1 - r^n)}{1 - r}$

◀ **Learn these results.**

These results are equivalent. Choosing the appropriate one reduces negative signs in calculations.

Example 4

A geometric series has first term 8 and common ratio 1.4. Find the sum of the first 10 terms.

Solution

Since $r > 1$, use the alternative form for the sum of a GP.

$$\begin{aligned} S_{10} &= \frac{a(r^{10} - 1)}{r - 1} = \frac{8(1.4^{10} - 1)}{1.4 - 1} \\ &= 558.51 \text{ (2 d.p.)} \end{aligned}$$

Graphical calculator support pack

Finding the number of terms in a geometric series involves solving an exponential equation. Recall from Chapter 7 that this can be done using logarithms.

Example 5

Find the number of terms in the geometric series $4 + 6 + 9 + \cdots + 30.375$.

Solution

Since the series is geometric the ratio of successive terms will give the common ratio.

$$r = \frac{u_3}{u_2} = \frac{9}{6} = 1.5$$

This means $a = 4$ and $r = 1.5$.

Now examine the last, or nth, term.

$$\begin{aligned} u_n &= ar^{n-1} \\ \Rightarrow \quad 30.375 &= 4 \times 1.5^{n-1} \\ \Rightarrow \tfrac{1}{4} \times 30.375 &= 1.5^{n-1} \\ \Rightarrow \quad 7.593\,75 &= 1.5^{n-1} \end{aligned}$$

Notice that the variable n now appears as a power.
Taking logarithms of both sides of the equation,

$$\begin{aligned} \log(7.593\,75) &= \log(1.5^{n-1}) \\ &= (n-1)\log(1.5) \end{aligned}$$

So $\quad n - 1 = \dfrac{\log(7.593\,75)}{\log(1.5)} = 5$

Recall that $\log a^n = n \log a$.

That is, $n = 6$
So the series $4 + 6 + 9 + \cdots + 30.375$ has six terms.

8.3 Geometric Progression
Exercise
Technique

1 Find (i) the next two terms and (ii) the nth term of the following geometric progressions:

a $1, 2, 4, 8, \ldots$
b $2, 6, 18, 54, \ldots$
c $4, 6, 9, 13\frac{1}{2}, \ldots$
d $0.1, 0.3, 0.9, 2.7, \ldots$

2 Given the geometric progressions $2, 10, 50, \ldots, u_n$, $48, 24, 12, \ldots, v_n$, and $1, 3, 9, 27, \ldots, w_n$, find:

a u_5
b u_8
c v_7
d v_{10}
e w_6
f w_9

3 Given the first term and common ratio of a geometric series, find the sum of the terms indicated:

a $a = 2, r = 2$: S_6
b $a = 10, r = -3$: S_7
c $a = 8, r = \frac{1}{2}$: S_6
d $a = -10, r = -\frac{4}{5}$: S_8

3 c

4 Find the sums of the following geometric series:

a $2 + 4 + 8 + \cdots + 256$
b $2 - 6 + 18 - 54 + \cdots + 13\,122$
c $16 + 24 + 36 + \cdots + 182.25$
d $2 - 6 + 18 - 54 + \cdots - 39\,366$
e $20 - 30 + 45 - \cdots - 341.72$
f $0.1 + 0.5 + 2.5 + \cdots + 312.5$

Contextual

1 Find the ninth term of the geometric series $12 + 8 + 5\frac{1}{3} + 3\frac{5}{9} + \cdots$, correct to three significant figures.

2 Find the common ratio of a geometric series given that the third term is 6 and the seventh term is 486.

2

3 Find the sum of the first ten terms of a geometric series with common ratio 2 and first term $1\frac{1}{2}$.

4 Write down the first three terms of $\sum_{n=1}^{10} 25(\frac{4}{5})^n$. State clearly:

4

a the first term of the GP
b the common ratio
c the sum of this series.

5 Evaluate:

a $\sum_{n=1}^{5} 200(1.1)^n$
b $\sum_{n=2}^{6} 80(\frac{7}{8})^n$

8.4 Convergence, Divergence and Oscillation

Consider the following sequences.

$3, 5, 7, 9, 11, \ldots, u_n, \ldots$

$1, 2, 1, 2, 1, \ldots, v_n, \ldots$

$\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \frac{1}{14}, \ldots, w_n, \ldots$

As n increases, what happens to u_n, v_n and w_n?

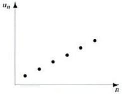
Notice that the first sequence is an AP with $a = 3$ and $d = 2$. The nth term, $u_n = 2n + 1$. As n gets large $(n \to \infty)$ u_n gets large $(u_n \to \infty)$. This sequence is said to **diverge**.

The second sequence simply oscillates between 1, when n is odd, and 2, when n is even. This sequence is said to be **oscillating**. A sequence that repeats itself in a regular pattern is **periodic**.

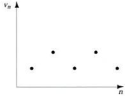
The third sequence is not an AP or GP. Check this by trying to find a common difference or common ratio. The sequence has nth term $w_n = \dfrac{1}{3n-1}$. As n gets large $(n \to \infty)$, w_n gets small $(w_n \to 0)$. This sequence is said to **converge**. The terms of the sequence get closer and closer to a limit of zero.

Verify this by considering the sequence formed by denominators only. Show that this is an AP with $a = 2$ and $d = 3$

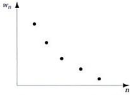
A graph can provide a good illustration of the behaviour of the sequence as n increases.



The sequence diverges to infinity. The values grow in size as n increases.



The sequence oscillates.



The sequence converges to a fixed value. In this case, $w_n \to 0$ as n increases.

Example 1

A sequence is defined by the recurrence relation $u_{n+1} = |6 - u_n|$. If $u_1 = 2$, describe the behaviour of the sequence.

Solution

Start by finding the first few terms of the sequence.

$$u_1 = 2$$

$$u_2 = |6 - u_1| = |6 - 2| = 4$$

$$u_3 = |6 - u_2| = |6 - 4| = 2$$

$$u_4 = |6 - u_3| = |6 - 2| = 4$$

The sequence is oscillating between 2 and 4. So $u_n = 2$ for odd n, and $u_n = 4$ for even n.

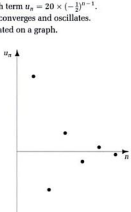
Example 2

Describe the behaviour of the GP $20, -10, 5, -2.5, 1.25, \ldots$.

Solution

This GP has $a = 20$ and $r = -\frac{1}{2}$. The nth term $u_n = 20 \times (-\frac{1}{2})^{n-1}$.
As n gets large, $u_n \to 0$. This sequence converges and oscillates.
The behaviour of this GP can be illustrated on a graph.

The behaviour of a sequence can be very important if the terms are being combined to form a series. This can determine if the series will diverge or converge as a large number of terms are added together. Adding an infinite number of terms of a series is known as a **sum to infinity**.



Example 3

Find the sum to infinity of the series $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \ldots$.

Solution

Notice that this series is created from a GP with $a = 1$ and $r = -\frac{1}{4}$. The nth term $u_n = 1 \times (-\frac{1}{4})^{n-1}$.

As n gets large, $u_n \to 0$. This means that the series increases by smaller numbers that oscillate between positive and negative.

To find the sum to infinity of the series write down the sum of the first n terms.

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{1(1-(-\frac{1}{4})^n)}{1+\frac{1}{4}} = \frac{4}{5}(1-(-\frac{1}{4})^n)$$

As $n \to \infty$, $(-\frac{1}{4})^n \to 0$.

This means that $S_{n\to\infty} = \frac{4}{5}$.

So $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots = \frac{4}{5}$.

This result can be generalised for any GP in which the common ratio r is such that $-1 < r < 1$.

This is often written mathematical shorth as $|r| < 1$.

The sum to infinity of a GP

GPs with common ratio r such that $|r| < 1$ generate convergent infinite series where $S_n \to S_\infty$ as $n \to \infty$, and

$$\boxed{S_\infty = \frac{a}{1-r}}$$

◀ **Learn this important result.**

Example 4

a A GP has first term 500 and common ratio $\frac{2}{5}$. What is the sum to infinity of the GP?

b The sum to infinity of a GP is 90. If the common ratio is $\frac{3}{5}$, what is the first term?

Solution

a $S_\infty = \dfrac{a}{1-r} = \dfrac{500}{1-\frac{2}{5}} = \dfrac{500}{\frac{3}{5}} = 833\frac{1}{3}$

b $S_\infty = \dfrac{a}{1-r} \Rightarrow 90 = \dfrac{a}{1-\frac{3}{5}} = \dfrac{a}{\frac{2}{5}}$

$$a = 90 \times \tfrac{2}{5} = 36$$

Series other than those generated by GPs can have a sum to infinity. These are often expressed using sigma notation and require the use of partial fractions (see Chapter 6).

Example 5

Find $\displaystyle\sum_{r=1}^{\infty} \frac{3}{(r+1)(r+2)}$

Solution

Substituting values of r in the sigma notation generates the following series.

$$\sum_{r=1}^{\infty}\frac{3}{(r+1)(r+2)}=\frac{3}{2\times 3}+\frac{3}{3\times 4}+\frac{3}{4\times 5}+\cdots$$
$$=\frac{3}{6}+\frac{3}{12}+\frac{3}{20}+\cdots$$
$$=\frac{1}{2}+\frac{1}{4}+\frac{3}{20}+\cdots$$

Notice that the generated sequence, $\frac{1}{2}, \frac{1}{4}, \frac{3}{20}, \ldots$, is neither an AP nor a GP. To sum this series, consider the expression for the nth term found from the sigma notation.

$$u_n=\frac{3}{(n+1)(n+2)}$$

This rational expression can be rewritten in partial fractions.

Recall from Chapter 6 that since the denominators are all linear factors the cover-up rule can be used.

$$u_n=\frac{3}{(n+1)(n+2)}\equiv\frac{3}{(n+1)}-\frac{3}{(n+2)}$$

Now return to the original series.

$$\sum_{r=1}^{\infty}\frac{3}{(r+1)(r+2)}\equiv\sum_{r=1}^{\infty}\left[\frac{3}{(r+1)}-\frac{3}{(r+2)}\right]$$

Substituting values of r from 1 to n, where n is a large number,

$$\sum_{r=1}^{n}\left[\frac{3}{(r+1)}-\frac{3}{(r+2)}\right]=\left(\frac{3}{2}-\frac{3}{3}\right)+\left(\frac{3}{3}-\frac{3}{4}\right)+\left(\frac{3}{4}-\frac{3}{5}\right)+\cdots$$
$$+\left(\frac{3}{n}-\frac{3}{n+1}\right)+\left(\frac{3}{n+1}-\frac{3}{n+2}\right)$$

Notice that, except for the first and last terms, all terms are repeated with one positive and one negative. These will cancel each other out.

$$\text{So }\sum_{r=1}^{n}\left[\frac{3}{(r+1)}-\frac{3}{(r+2)}\right]=\frac{3}{2}-\frac{3}{n+2}$$

As $n\to\infty$, $\frac{3}{n+2}\to 0$. So the series converges to the value $\frac{3}{2}$.

$$\sum_{r=1}^{\infty}\frac{3}{(r+1)(r+2)}=\frac{3}{2}$$

Check this by summing successive terms of the series using a calculator. You should find that the sum of the series never exceeds $\frac{3}{2}$.

8.4 Convergence, Divergence and Oscillation

Exercise

Technique

1 Find the sum to infinity for a geometric progression with:

a $a = 200, r = \frac{1}{2}$
b $a = 64, r = \frac{1}{8}$
c $a = 600, r = 0.6$
d $a = 30, r = \frac{4}{3}$
e $a = 200, r = 0.88$
f $a = 1000, r = 0.1$

2 Describe the behaviour of the following sequences:

a $u_{n+1} = |8 - u_n|$ where $u_1 = 10$
b $\cos(30n)$
c $\ln 8, (\ln 8)^2, (\ln 8)^3,$
d $\ln 1.8, (\ln 1.8)^2, (\ln 1.8)^3, \ldots$
e $u_n = \dfrac{5}{(n+2)(n+3)}$
f $u_n = \dfrac{10}{n(n+1)}$

3 Find the sums to infinity of the following geometric series:

a $243 + 81 + 27 + \ldots$
b $243 - 81 + 27 - \ldots$
c $12 + 4 + \frac{4}{3} + \ldots$
d $100 + 80 + 64 + \ldots$
e $20 - 8 + 3.2 - \ldots$
f $20 - 18 + 16.2 - \ldots$

4 For the following geometric progressions find the values of the unknown terms:

a $r = \frac{1}{4}, S_\infty = 40, a = ?$
b $a = 3, S_\infty = 8, r = ?$
c $r = \frac{3}{5}, S_\infty = 45, a = ?$
d $a = 10, S_\infty = 30, r = ?$
e $a = 45, S_\infty = 50, r = ?$
f $r = -\frac{7}{8}, S_\infty = 16, a = ?$

Contextual

1 Find the sum to infinity of the geometric series with first term 15 and common ratio $\frac{3}{4}$.

2 The first term of a geometric series is 24 and the common ratio is $-\frac{1}{2}$. Find the sum to infinity.

3 The sum to infinity of a GP is 80 and the first term is 16. Find the common ratio.

4 Given that the common ratio of a geometric series is $\frac{3}{8}$ and the sum to infinity is 32, find the first term.

5 The sum to infinity of a geometric progression is ten times the first term. Find the common ratio.

6 The sum to infinity of a GP is 81 and the sum of the first four terms is 65. Find the first term and the common ratio.

7 Gerry says, 'The sum to infinity of a GP with first term 12 and common ratio $\frac{4}{3}$ is 48'. John immediately says that it must be wrong.

a Explain why John is correct.

b Given that the sum to infinity and the first term are correct, find the common ratio.

8.5 The Binomial Theorem and Power Series

What does 'binomial' mean? 'Bi' is Latin, meaning 'double' so a binomial is an expression with two terms. Some examples of binomials are $(a+b)^2$, $(3+x)^4$, and $(5-y)^{\frac{1}{2}}$.
The binomial $(a+b)^2$ can be rewritten as a sum of algebraic terms.

$$(a+b)^2 = a^2 + 2ab + b^2$$

There are also algebraic expressions for binomials with powers greater than 2

$$\begin{aligned}
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= (a+b)(a+b)^3 \\
&= (a+b)(a^3 + 3a^2b + 3ab^2 + b^3) \\
&= a(a^3 + 3a^2b + 3ab^2 + b^3) + b(a^3 + 3a^2b + 3ab^2 + b^3) \\
&= a^4 + 3a^3b + 3a^2b^2 + ab^3 + ba^3 + 3a^2b^2 + 3ab^3 + b^4 \\
&= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{aligned}$$

Recall from Chapter
that 'polynomial' is
Greek for 'many term

Consider the following structure:

$$\begin{aligned}
(a+b)^1 &= a + b \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{aligned}$$

Now write down the coefficients only of the terms in these expansions. As they are written down arrange them in a triangular form under each other.

$$\begin{array}{ccccccccc}
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Blaise Pascal (1623–1662)
Pascal, a French mathematician, is credited with a patte formed by the coefficients in thes expansions. It is usually called Pasca triangle.

Write down the next row in this triangle. Check that each figure is created by adding the numbers directly above it.

$$\begin{array}{ccccccccccc}
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

So $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Notice:

- the position of the coefficients from **Pascal's triangle**
- that the powers of a decrease from 5 to 0 and the powers of b increase from 0 to 5
- that adding the powers of a and b in any term gives 5, which is the power on the original binomial.

It is useful to remember these points when expanding a binomial.

Example 1

Expand:

a $(2+x)^4$ **b** $(3-x)^5$

Solution

a Notice that the index, or power, here is 4. The fourth line of Pascal's triangle is

1 4 6 4 1

These will be the coefficients of the terms in the expansion. Build up the expansion in two steps.
Insert powers of the first term, '2'.

$1 \times 2^4 \quad 4 \times 2^3 \quad 6 \times 2^2 \quad 4 \times 2^1 \quad 1 \times 2^0$

Insert powers of the second term, 'x'.

$1 \times 2^4 \times x^0 \quad 4 \times 2^3 \times x^1 \quad 6 \times 2^2 \times x^2 \quad 4 \times 2^1 \times x^3 \quad 1 \times 2^0 \times x^4$

Now evaluate any numbers raised to a power, recalling that $a^0 = 1$ for all values of a. The expansion now becomes

$(2+x)^4 = 16 + 32x + 24x^2 + 8x^3 + x^4$

b The power here is 5. The fifth line of Pascal's triangle is required.

1 5 10 10 5 1

Insert powers of the first term, '3'.

$1 \times 3^5 \quad 5 \times 3^4 \quad 10 \times 3^3 \quad 10 \times 3^2 \quad 5 \times 3^1 \quad 1 \times 3^0$

Insert powers of the second term, '$-x$'.

Remember to include the negative sign.

$1 \times 3^5 \times (-x)^0 \quad 5 \times 3^4 \times (-x)^1 \quad 10 \times 3^3 \times (-x)^2$

$10 \times 3^2 \times (-x)^3 \quad 5 \times 3^1 \times (-x)^4 \quad 1 \times 3^0 \times (-x)^5$

Even powers of $(-x)$ are positive and odd powers of $(-x)$ are negative, so

$$(3-x)^5 = 243 - 405x + 270x^2 - 90x^3 + 15x^4 - x^5$$

This technique can be used to expand $(a+x)^n$ or $(a-x)^n$ for any positive integer n. The problem then becomes finding the values of the coefficients in the nth row of Pascal's triangle.

Factorial notation

A **factorial** is a product of consecutive natural numbers, starting at 1.
So 3 factorial, written $3! = 3 \times 2 \times 1$ and $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$
A table of factorial totals shows how concise this notation can be.

$1! = 1$	$3! = 6$	$5! = 120$	$7! = 5040$	$9! = 362\,880$
$2! = 2$	$4! = 24$	$6! = 720$	$8! = 40\,320$	$10! = 3\,628\,800$

How do factorials fit with the coefficients in Pascal's triangle? The terms of Pascal's triangle are known also as **binomial coefficients**. The third coefficient on the fourth row can be written using factorials as

$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{24}{6 \times 1} = 4$$

Check this result in Pascal's triangle.

In a similar way the rth coefficient on the nth row of Pascal's triangle can be written as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

◀ **Learn this important result.**

Remember that r star at 0, not 1, and that $0! = 1$ by definition.

The general result for the expansion of $(a+x)^n$ is:

$$(a+x)^n = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \cdots + \binom{n}{r}a^{n-r}x^r + \cdots + x^n$$

◀ **Learn this result.**

Check how the powe of a and x relate to th terms inside the binomial coefficient.

Example 2

Find the values of:

a $\binom{6}{4}$ **b** $\binom{7}{2}$

Solution

a $\binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6!}{4!2!} = \frac{720}{24 \times 2} = 15$

b $\binom{7}{2} = \frac{7!}{2!(7-2)!} = \frac{7!}{2!5!} = \frac{5040}{2 \times 120} = 21$

To calculate the value of binomial coefficients not all factorials need be remembered. The fraction can be cancelled down quickly since each factorial is a product of numbers beginning at 1.

Example 3

Without using a calculator, find $\binom{8}{3}$.

Solution

$$\binom{8}{3} = \frac{8!}{3!(8-3)!} = \frac{8!}{3!5!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (5 \times 4 \times 3 \times 2 \times 1)}$$

Notice that 5! is a common factor of both the denominator and the numerator. This means it can be cancelled.

$$\binom{8}{3} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1}$$

Now look for other common factors.

$$\binom{8}{3} = \frac{8 \times 7}{1} = 56$$

The 3×2 in the denominator eliminates the 6 in the numerator.

One advantage of using factorial notation is that particular terms in an expansion can be found without finding the whole series.

Example 4

Find the term involving x^9 in the expansion of $(1 + 2x)^{12}$.

Solution

Recall the general result for expanding $(a + x)^n$.

$$(a + x)^n = a^n + \cdots + \binom{n}{r} a^{n-r} x^r + \cdots + x^n$$

So the term involving x^9 in this expansion will be

$$\binom{12}{9} 1^3 (2x)^9$$

$$\binom{12}{9} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220$$

Notice that $n = 12$, $r = 9$ and $a = 1$, and x has a multiplier (coefficient) of 2

So the term in x^9 is then $220 \times 1^3 \times 2^9 x^9 = 112\,640x^9$

Notice that this provides a quick method of identifying terms in the series.

The expansion of binomials using factorial notation for the binomial coefficients is often stated as a mathematical theorem.

The binomial theorem

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$
$$= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b$$
$$+ \cdots + \binom{n}{n-1} ab^{n-1} + \binom{n}{n} b^n$$

◀ **Learn this result.**

Notice that $\binom{n}{0}$ and $\binom{n}{n}$ both involve 0!.

The expansion of the brackets can be written as a series using sigma notation. The binomial coefficients relate clearly to the powers of the terms a and b. The case where $a = 1$ and b is the variable x is particularly interesting.

This series was discovered by Ne in 1676 and correctl derived by Euler abou 100 years later.

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots$$

This can be used to find the approximate value of powers of numbers close to 1.

Example 5

a Find an approximation correct to four decimal places of $(1.01)^6$.

b Find $(0.99)^4$ without using a calculator.

Solution

a This approximation can be made by using the binomial series for $(1+x)^6$

$$(1+x)^6 = 1 + \binom{6}{1}x + \binom{6}{2}x^2 + \binom{6}{3}x^3 + \cdots$$
$$= 1 + 6x + 15x^2 + 20x^3 + \cdots$$

Now write 1.01 as $(1 + 0.01)$. This is better expressed as $(1 + \frac{1}{100})$. Substitute $x = \frac{1}{100}$ into the series expansion.

$$(1.01)^6 = 1 + \frac{6}{100} + \frac{15}{100^2} + \frac{20}{100^3} + \cdots$$
$$= 1 + 0.06 + 0.0015 + 0.000\,02 + \cdots$$
$$= 1.061\,52$$

Notice that successiv terms are so small th they will not influen the first four decimal places.

So, to four decimal places, an approximation for $(1.01)^6$ is 1.0615.

b Since $0.99 = 1 - 0.01$, $(0.99)^4$ can be rewritten as $(1 - 0.01)^4$. Use the binomial series for $(1-x)^4$ with $x = \frac{1}{100}$

$$(1-x)^4 = 1 + \binom{4}{1}(-x) + \binom{4}{2}(-x)^2 + \binom{4}{3}(-x)^3 + (-x)^4$$
$$= 1 - 4x + 6x^2 - 4x^3 + x^4$$

$$\left(1-\frac{1}{100}\right)^4 = 1 - \frac{4}{100} + \frac{6}{100^2} - \frac{4}{100^3} + \frac{1}{100^4}$$
$$= 1 - 0.04 + 0.0006 - 0.000\,004 + 0.000\,000\,01$$
$$= 0.960\,596\,01$$

Power series

A power series is a polynomial. The terms of the series are powers of a variable, usually x, multiplied by a coefficient.

$$a_0 + a_1x + a_2x^2 + \cdots = \sum_{r=1}^{n} a_r x^r$$

Power series can be **divergent** or **convergent**. When the series is convergent there is usually a restriction on the range of values of the variable x.

A power series can be created from a binomial expansion where the power is fractional and not an integer. The expansion is now written slightly differently to how it was before, because the binomial coefficient, $\binom{n}{r}$, is nonsensical when n is a fraction.

When n is negative or fractional, the binomial series for $(1+x)^n$ is given by

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots$$

◀ Learn this result.

Notice that this series will be infinite if n is negative or fractional because the numerator will never contain the factor $(n-n)$.

This power series will converge provided $-1 < x < 1$. This condition is particularly important because it means that the series is not valid for all values of x.

Example 6

Find the first five terms in the binomial expansion of $\frac{1}{1-x}$ and state the range of values for which it is valid.

Solution

Applying the law of indices, $\frac{1}{1-x} = (1-x)^{-1}$. This is a binomial with power -1. The series can be found by substituting $n = -1$ into the binomial expansion (carefully checking the sign of x).

Recall that $\frac{1}{a} = a^{-1}$.

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots$$

$$\text{So } (1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)(-x)^2}{2!} + \frac{(-1)(-2)(-3)(-x)^3}{3!}$$
$$+ \frac{(-1)(-2)(-3)(-4)(-x)^4}{4!} + \cdots$$

$$= 1 + x + \frac{2x^2}{2} + \frac{6x^3}{6} + \frac{24x^4}{24} + \cdots$$
$$= 1 + x + x^2 + x^3 + x^4 + \cdots$$

This series is valid for $-1 < x < 1$; that is, $|x| < 1$.

Check what happens when $x = \frac{1}{2}$ and $x = 2$. Notice the need for the restriction $|x| < 1$.

Example 7

Find the first four terms of the binomial series for $\sqrt{1+x}$. State the range of values of x for which the series converges.

Solution

$\sqrt{1+x} = (1+x)^{\frac{1}{2}}$. This is a binomial, with power $\frac{1}{2}$.

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots$$

$$\text{So } (1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})x^2}{2!} + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})x^3}{3!} + \cdots$$
$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{3}{48}x^3 - \cdots$$
$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \cdots$$

This series is valid for $|x| < 1$.

The binomial series can be used to provide a power series expansion for rational functions.

Example 8

Write $\dfrac{1-x}{(1+2x)^4}$ as a series of ascending powers of x, up to and including the term in x^3. State the range of values of x for which the series is valid.

Solution

Once the rational function is written as a product the application of the binomial theorem can be seen.

Expand $(1+2x)^{-4}$ as a power series, then multiply this result by $(1-x)$.

$$\text{Check that } (1+2x)^{-4} = 1 - 8x + 40x^2 - 160x^3 + \cdots$$
$$\text{Then } (1-x)(1+2x)^{-4} = (1-x)(1 - 8x + 40x^2 - 160x^3 + \cdots)$$
$$= 1 - 9x + 48x^2 - 200x^3 + \cdots$$

This series is valid for $-1 < 2x < 1$; that is $|x| < \frac{1}{2}$.

8.5 The Binomial Theorem and Power Series

Exercise

Technique

1 Find the expansion of the following binomials using Pascal's triangle:

a $(1+x)^6$ b $(3-x)^4$ c $(5-x)^4$ d $(2+x)^5$

1 d

2 Find the values of the following binomial coefficients:

a $\binom{4}{2}$ b $\binom{7}{4}$ c $\binom{5}{3}$ d $\binom{20}{18}$

3 Expand the following using the binomial theorem:

a $(1+x)^7$ b $(4-x)^4$ c $(2-3x)^6$ d $(x-\frac{3}{x})^4$

4 Write the following as series of ascending powers of x, up to and including the term in x^3:

a $(1-x)^{20}$ b $(3-x)^5$ c $(y+x)^7$ d $(2-3x)^5$

5 Find the range of values of x for which the series expansions of the following are valid:

a $(1+x)^{\frac{1}{2}}$ b $(3-2x)^{\frac{1}{3}}$ c $\sqrt[3]{1+\frac{1}{5}x}$ d $(4-3x)^{-10}$

6 Find the first four terms of the binomial series for the following functions:

a $(1+x)^{-8}$ b $\sqrt[3]{1+2x}$ c $\sqrt{1-2x}$ d $\dfrac{1+x^2}{(1-x)^4}$

6 c

Contextual

1 Find the binomial expansion of $(4+x)^6$. When is this expansion valid?

2 Find the term involving x^9 in the expansion of $(1+2x)^{11}$.

3 Find the coefficient of the x^7 term in the expansion of $(1-3x)^{15}$.

4 The expansion of $(x-\frac{3}{x})^{12}$ has a term that does not contain x. State the value of this term.

5 Find the value of $(1.01)^7$ to four decimal places using the binomial expansion of $(1+x)^n$.

6 Expand $\sqrt{1-4x}$ as a series of ascending powers of x, up to and including the term in x^3. State the range of values of x for which the expansion is valid.

Consolidation

Exercise A

1 Find the sum of the arithmetic series $2 + 5 + 8 + \cdots + 398$.

(NEAB)

2 The tenth term of the arithmetic progression is zero and the sum of the first 10 terms is 15.

a Find the first term and the common difference.

b How many more terms must be added for the sum of the arithmetic progression to be zero?

(OCSEB)

3 A geometric series has third term 27 and sixth term 8.

a Show that the common ratio of the series is $\frac{2}{3}$.

b Find the first term of the series.

c Find the sum to infinity of the series.

d Find, to three significant figures, the difference between the sum of the first 10 terms of the series and the sum to infinity of the series.

(ULEAC)

4 Find the sum of the series with first term 11 and last term 40, in which each term, after the first, exceeds the previous term by $\frac{1}{2}$.

(UCLES)

5 The nth term of a sequence is u_n, where $u_n = 95(\frac{4}{5})^n$, $n = 1, 2, 3, \ldots$.

a Find the values of u_1 and u_2.

Giving your answers to three significant figures, calculate:

b the value of u_{21}

c $\sum_{n=1}^{15} u_n$

d Find the sum to infinity of the series whose first term is u_1 and whose nth term is u_n.

(ULEAC)

6 Find, in the simplest form, the first three terms in the expansion of $(1 + 3t)^{\frac{2}{3}}$, in ascending powers of t, where $|t| < \frac{1}{3}$.

(NEAB)

7 Two sequences $u_1, u_2, u_3, \ldots$ are defined as follows:

Sequence A: $u_1 = 2 \quad u_{n+1} = 3 - u_n$

Sequence B: $u_1 = 2 \quad u_{n+1} = u_n + \dfrac{1}{2^n}$

a For each of the two sequences, find the values of u_2, u_3, u_4, and u_5.
b State for each sequence whether it is convergent, divergent or oscillating.

(NEAB)

8 Given that $|x| < \frac{1}{2}$, expand $\sqrt{1+2x}$ as a series of ascending powers of x, up to and including the term in x^3, simplifying the coefficients.

(UCLES)

9 Find the term in x^{12} in the binomial expansion of $(x+2x^2)^{10}$.

(OCSEB)

10 a Use the binomial theorem to give the expansion of $(x+y)^4$.
b Hence obtain the expansion of $(x+\frac{2}{x})^4$, expressing each term in a simplified form.
c Find the coefficient of x^2 in the expansion of $(1+x^2)(x+\frac{2}{x})^4$.

(NEAB)

Exercise B

1 Find the sum of the arithmetic series $3+7+11+\cdots+79$.

2 Helen's father gives her a loan of £10 800 to buy a car. The loan is to be repaid by 12 unequal monthly instalments, starting with an initial payment of £A in the first month. There are no interest charges on the remaining debt but the instalments increase by £60 per month so that the second monthly payment is £$(A+60)$, the third monthly payment is £$(A+120)$ and so on.

a Show that $A = 570$.
b Find an expression in terms of n for the remaining debt immediately after Helen makes her nth payment. Give your answer in a fully factorised form.

(NEAB)

3 a i Express $\dfrac{8}{(2r-1)(2r+3)}$ in partial fractions.
ii Hence find the sum of n terms of the series

$$\frac{8}{1\times5}+\frac{8}{3\times7}+\frac{8}{5\times9}+\cdots$$

b A sequence of integers u_1, u_2, u_3 is defined by $u_1 = 5$ and $u_{n+1} = 3u_n - 2^n$ for $n \geq 1$. Use this definition to find u_2 and u_3.

(OCSEB)

4 **a** The sixth term of a geometric progression is 6.075; the fifth term is 4.05. Calculate:

i the common ratio

ii the first term

iii the 30th term (correct to three significant figures).

b State, with a reason, whether the geometric series

$$\ln 3 + (\ln 3)^2 + (\ln 3)^3 + \cdots$$

is divergent or convergent.

c The series S is given by

$$\ln 3 + \ln(3^2) + \ln(3^3) + \cdots + \ln(3^{30}) = \sum_{r=1}^{30} \ln(3^r)$$

Show that S is an arithmetic series whose sum is approximately 511.

d Show that $\sum_{r=1}^{n} \ln(3r) = n \ln 3 + \ln(n!)$.

(OCSEB)

5 **a** The first term of an arithmetic progression is 9 and the seventh term is three times the second term. Find the common difference and the sum of the first 32 terms of the arithmetic progression.

b The first term of a geometric progression is 81 and the fourth term is 24. Find the common ratio and the sum to infinity of the geometric progression.

(WJEC)

6 A sequence is defined inductively as follows:

$$u_1 = 2$$

$u_{n+1} =$ units digit of $2u_n$ (that is, the remainder when $2u_n$ is divided by 10)

a Write down the values of u_i for $i = 1, 2, 3, 4, 5$ and show that $u_6 = 4$.

b State whether or not this sequence is periodic or convergent

c Find the units digit of 2^{222}, explaining how you obtain your answer.

(OCSEB)

7 The nth term, u_n, of four sequences is defined below. For each sequence decide whether it is convergent, divergent to $+\infty$, divergent to $-\infty$ or oscillating. For each convergent sequence, state the limit to which it tends as $n \to \infty$.

a $u_n = 2 + \sqrt{n}$

b $u_n = 5 - \dfrac{1}{n^2}$

c $u_n = \sin(\frac{1}{2}n\pi)$

d $u_n = \dfrac{3n}{1+n}$

(AEB)

Applications and Activities

Loan repayments

The sum of a geometric series can be used to find:

- the time taken to repay a loan given a debt, interest rate and repayment instalments;
- the repayment instalment given a debt, interest rate and repayment term.

1 You borrow £1000 at 14% APR and pay back £300 per year. How long will it take to repay the debt?

2 You borrow £1000 at 14% APR and want to repay the debt in three years. What should your monthly instalment be?

3 Investigate credit terms for finance companies and check the repayment tables for the quoted APR.

APR is the Annual Percentage Rate and must, by law, be quoted on all credit arrangements.

Summary

- A **sequence** is a set of numbers occurring in order. When the terms of a sequence are added together, a **series** is formed.
- A sequence can be described using an algebraic term for u_n or by a **recurrence relation**.
- A series can be written using **sigma notation**, which uses the Greek letter Σ to represent summation over a number of terms.
- The nth term of an AP is $\boldsymbol{a + (n-1)d}$, where a is the first term, and d is the **common difference**.
- The **sum to** n **terms** of an arithmetic series can be written in equivalent forms

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_n = \frac{n(a+l)}{2}$$ where l is the last term.

- The nth term of a GP is $\boldsymbol{ar^{n-1}}$ where a is the first term and r is the **common ratio**.

- The sum to n terms of a geometric series with first term a and **common ratio** r is given by

$$S_n = \frac{a(1-r^n)}{1-r}$$

- A sequence is said to **diverge** if $u_n \to \infty$ as $n \to \infty$, **converge** if $u_n \to$ a limit as $n \to \infty$, and be **periodic** if it repeats itself in a regular pattern.
- GPs with a common ratio r such that $|r| < 1$ create convergent infinite series where $S_n \to S_\infty$ as $n \to \infty$, and

$$S_\infty = \frac{a}{1-r}$$

- The coefficients in a binomial expansion, $(a+b)^n$, can be written down using Pascal's triangle.
- A **factorial** is a product of consecutive natural numbers; so $3! = 3 \times 2 \times 1$ and $0! = 1$ by definition.
- The binomial theorem states that, for positive integral values of n ($n \in \mathbb{Z}^+$) says:

$$\begin{aligned}(a+b)^n &= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \\ &= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b \\ &\quad + \cdots + \binom{n}{n-1} ab^{n-1} + \binom{n}{n} b^n\end{aligned}$$

- The binomial theorem can be used to write a binomial series for integer, negative and rational powers. This power series will have a condition on the values for which it is convergent.

9 Integration I

What you need to know

- How to differentiate standard functions.
- How to use the laws of indices.
- How to find the value of an algebraic expression.
- How to sketch graphs.
- How to use sigma notation.

Review

1 Find:

a $\dfrac{d}{dx}(x^2 + 1)$

b $\dfrac{d}{dx}(\frac{1}{2}x^2 - 6x + 2)$

c $\dfrac{d}{dx}\left(x^{\frac{1}{2}} - \dfrac{1}{x}\right)$

d $\dfrac{d}{dx}(\cos x)$

e $\dfrac{d}{dx}(\sin 2x)$

f $\dfrac{d}{dx}(2 - x^2 + 3\cos 5x)$

2 Write each of the following expressions as Ax^n or $(Ax^n + Bx^m)$, where A and B are constants.

a $\dfrac{1}{x^2}$ b $\dfrac{3x^2}{x^5}$ c $\dfrac{7x - x^2}{x^5}$

d $\dfrac{2x}{x^{\frac{1}{2}}}$ e $\dfrac{5}{\sqrt{x}}$ f $\sqrt{x}(1 + \sqrt{x})$

3 Find the value of each of the following expressions for the given value of x:

a $3x^3 + 5x^2 + x$, when $x = 2$

b $\frac{1}{4}x^4 - x^3 - x$, when $x = -2$

c $2x^{\frac{1}{2}} - x^{-\frac{1}{2}} + 1$, when $x = 9$

d $x^{\frac{2}{3}}$, when $x = 8$

4 Sketch the graphs of:

a $y = 1$ b $y = x^2$ c $y = x^3$

d $y = \sin x$ e $y = 1 + \sin x$ f $y = 2\cos x$

5 Simplify the following using sigma notation:

a $1 + 2 + 3$

b $1 + 4 + 9 + 16 + 25$

c $2 + 4 + 6 + 8$

d $4 + 7 + 10 + 13$

9.1 Indefinite Integration

Integration can be described as a reverse operation. Consider some familiar reverse operations.

Operation	Reverse operation	Example
Add	Subtract	$7 \xrightarrow{+3} 10$, $10 \xrightarrow{-3} 7$
Multiply	Divide	$12 \xrightarrow{\times 3} 36$, $36 \xrightarrow{\div 3} 12$
Cube	Cube-root	$2 \xrightarrow{\text{cube}} 8$, $8 \xrightarrow{\text{cube root}} 2$

Mathematicians studying what we now call differentiation and integration realised that these processes were closely related. Leibniz (1646–1716) and Isaac Newton (1643–1727) independently discovered that integration is the reverse of differentiation. This fact is called the **fundamental theorem of calculus.**

Operation	Reverse operation	Example
Differentiation	Integration	$x^n \xrightarrow{\text{differentiate: multiply by } n \text{, reduce power by 1}} nx^{n-1}$, $nx^{n-1} \xrightarrow{\text{integrate: add 1 to the power, divide by this number}} x^n$

The notation used in differentiation a
integration is based
notation developed b
Leibniz. He was a
seventeenth century
mathematician, philo
and statesman. He
published his work or
calculus in 1684.

The principle of the reverse process will be used to integrate powers of x by increasing the power by 1 and then dividing the new term by this number – the reverse of differentiation.

$$\int x^2 \, \mathrm{d}x = \tfrac{1}{3}x^3$$ ◀ **Differentiating $\frac{1}{3}x^3$ gives x^2.**

Check this by differentiating the answer; it should reverse the integration process.

But note that $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{3}x^3 + 1) = 3 \times \frac{1}{3}x^2 + 0 = x^2$

and $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{3}x^3 + 2) = 3 \times \frac{1}{3}x^2 + 0 = x^2$

and so on.

Notation
$\int$ is the symbol for 'integrate'
The term that is bein
integrated is the **integrand**.
'dx' means integrate 'with respect to' the variable x
So $\int f(x)\,\mathrm{d}x$ means integrate the integra
$f(x)$ with respect to

This means that any of the expressions $\frac{1}{3}x^3, \frac{1}{3}x^3+1, \frac{1}{3}x^3+2, \ldots$ could be the answer to $\int x^2 dx$. Differentiation of each of these expressions gives x^2. Since there is no definite answer the expression $\int x^2 dx$ is called an **indefinite integral**. The complete answer is

$$\int x^2 dx = \frac{1}{3}x^3 + c, \text{ where } c \text{ is a constant.}$$

If the variable was t, not x, we would 'integrate t^2 with respect to t'; $\int t^2\, dt = \frac{1}{3}t^3 + c$.

All indefinite integrals have '$+c$' at the end where c is a constant.
The following table illustrates the basic principles of integration.

$\int f(x)dx$	$g(x)$	Check $g'(x) = f(x)$	In general:
$\int dx = \int 1\, dx$ $\int 5dx$	$x + c$ $5x + c$	$\frac{d}{dx}(x+c) = 1 + 0 = 1$ $\frac{d}{dx}(5x+c) = 5 + 0 = 5$	$\int k\, dx = kx + c$, where k is any number
$\int x\, dx$ $\int 7x\, dx$	$\frac{1}{2}x^2 + c$ $\frac{7}{2}x^2 + c$	$\frac{d}{dx}(\frac{1}{2}x^2 + c) = \frac{1}{2} \times 2x + 0 = x$ $\frac{d}{dx}(\frac{7}{2}x^2 + c) = \frac{7}{2} \times 2x + 0 = 7x$	$\int kf(x)\, dx = k\int f(x)\, dx$, where k is any constant
$\int x^2\, dx$ $\int x^3\, dx$ $\int x^{\frac{1}{2}} dx$	$\frac{1}{3}x^3 + c$ $\frac{1}{4}x^4 + c$ $\frac{2}{3}x^{3/2} + c$	$\frac{d}{dx}(\frac{1}{3}x^3 + c) = \frac{1}{3} \times 3x^2 + 0 = x^2$ $\frac{d}{dx}(\frac{1}{4}x^4 + c) = \frac{1}{4} \times 4x^3 + 0 = x^3$ $\frac{d}{dx}(\frac{2}{3}x^{3/2} + c) = \frac{2}{3} \times \frac{3}{2}x^{1/2} + 0 = x^{1/2}$	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$ ◀ True for all n, except for the special case $n = -1$ (see Chapter 12).
$\int (2x^2 - x + 6)\, dx$	$\frac{2}{3}x^3 - \frac{1}{2}x^2 + 6x + c$	$\frac{d}{dx}(\frac{2}{3}x^3 - \frac{1}{2}x^2 + 6x + c) =$ $\frac{2}{3} \times 3x^2 - \frac{1}{2} \times 2x + 6 + 0 =$ $2x^2 - x + 6$	$\int (f(x) + g(x))\, dx =$ $\int f(x)\, dx + \int g(x)\, dx$

Recall that $g'(x)$ is alternative notation for $\frac{d}{dx}g(x)$.

The constant can be 'pulled outside' the integral sign.

$\int x^{1/2}\, dx = \frac{1}{\frac{3}{2}}x^{3/2} + c$; to divide by a fraction, invert it and multiply, to give $\frac{2}{3}x^{3/2} + c$.

Integrate each term separately.

Summary of the basic principles

- $\int k\, dx = kx + c$, where k is any number
- $\int kf(x)\, dx = k\int f(x) dx$, where k is any number
- $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$, where n is any number except -1
- $\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x) dx$

Example 1

Find:

a $\int x(x-3)dx$ **b** $\int (1 - 3\sqrt{x})dx$ **c** $\int \left(\frac{3x^2 - 1}{x^2}\right)dx$

Solution

a The technique here is to expand the brackets first, and then integrate each term separately.

$$\int x(x-3)\, dx = \int (x^2 - 3x)\, dx$$
$$= \tfrac{1}{3}x^3 - 3 \times \tfrac{1}{2}x^2 + c$$
$$= \tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + c$$

◀ Always add a constant for an indefinite integral.

Use $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$.

b To integrate the square root, first express it in powers of x.

$$\int(1-3\sqrt{x})\,dx = \int(1-3x^{\frac{1}{2}})\,dx$$

$$= x - 3 \times \frac{1}{\frac{3}{2}}x^{\frac{3}{2}} + c$$

$$= x - 2x^{\frac{3}{2}} + c$$

◀ **Check the answer by differentiation.**

Recall that $\sqrt{x} = x^{\frac{1}{2}}$.

Use $\int x^n\,dx = \frac{x^{n+1}}{n+1}$ +

To divide by a fracti invert and multiply.

c First express the integrand as powers of x.

$$\int\left(\frac{3x^2-1}{x^2}\right)dx = \int\left(\frac{3x^2}{x^2} - \frac{1}{x^2}\right)dx$$

$$= \int\left(3 - \frac{1}{x^2}\right)dx$$

$$= \int(3 - x^{-2})\,dx$$

$$= 3x - \frac{x^{-1}}{-1} + c$$

$$= 3x + \frac{1}{x} + c$$

◀ **Remember to add a constant for an indefinite integral.**

Note the common denominator.

Use $\int x^n\,dx = \frac{x^{n+1}}{n+1}$ +

Example 2

The gradient of a curve $y = f(x)$ is given by $\frac{dy}{dx} = (2-x)(2-3x)$. Find the equation of the curve, given that it passes through the point $(3, 2)$.

Solution

To find y from $\frac{dy}{dx}$, reverse the differentiation by using integration.

$$y = \int(2-x)(2-3x)dx$$

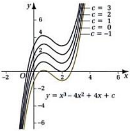
$$= \int(4 - 8x + 3x^2)dx$$

$$= 4x - \frac{8x^2}{2} + \frac{3x^3}{3} + c$$

$$y = 4x - 4x^2 + x^3 + c$$

To integrate, expand brackets and integra each term separately

This equation represents a whole family of curves. The graphs can be drawn on a graphical calculator using different values of c. The graphs show that there are many functions with a gradient of $(2-x)(2-3x)$. However, there is only one graph that passes through the point $(3, 2)$. Check that the value of c must be -1.



An alternative to the graphical method is to use algebra to find, or check, the value of c. Substituting the values of $x = 3$ and $y = 2$ into the expression for y,

$$y = 4x - 4x^2 + x^3 + c$$
$$\Rightarrow \quad 2 = 4 \times 3 - 4 \times 9 + 27 + c$$
$$\Rightarrow \quad 2 = 12 - 36 + 27 + c$$
$$\Rightarrow \quad -1 = c$$

So the equation of the curve is $y = x^3 - 4x^2 + 4x - 1$.

Integration and mechanics

In Chapter 5, mechanics was used as a context for differentiation. Recall that if s = distance travelled in time t then the velocity is given by $v = \frac{ds}{dt}$, and the acceleration is given by $a = \frac{dv}{dt}$. Using integration as the reverse process to differentiation it is now possible to work backwards. This means that we can find v if we know a, and can find s if we know v. The new equations become:

$$v = \int a \, dt \quad \text{and} \quad s = \int v \, dt.$$

Example 3

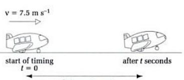
An aircraft taxies at a constant speed of 7.5 m s^{-1} to the start of a straight runway. It then begins to accelerate. Its acceleration after t seconds is given by $a = 2t - \frac{1}{5}t^2$.

a Find an expression for the velocity at time t seconds.

b An observer estimates that 10 seconds after leaving the start of the runway the plane takes off. What distance has it travelled along the runway in that time?

Solution

Draw a diagram to illustrate the situation.



a $a = \frac{dv}{dt} = 2t - \frac{1}{5}t^2$

We know that $v = \int a\,dt$, so we integrate the acceleration to find the velocity.

$v = \int a\,dt$

$= \int (2t - \frac{1}{5}t^2)\,dt$

$= 2 \times \frac{1}{2}t^2 - \frac{1}{5} \times \frac{1}{3}t^3 + c$

$= t^2 - \frac{1}{15}t^3 + c$

The variable is t.

$\int t^n\,dt = \frac{t^{n+1}}{n+1} + c$

There is enough information in the question to calculate c. At the start of the straight runway the speed of the aircraft is $7.5\,\mathrm{m\,s^{-1}}$. So when $t = 0$, $v = 7.5$. Substitute these values into the expression for v.

$7.5 = 0 - 0 + c$

$\frac{15}{2} = c$

$v = t^2 - \frac{1}{15}t^3 + \frac{15}{2}$

b $v = \frac{ds}{dt} = t^2 - \frac{1}{15}t^3 + \frac{15}{2}$

So $s = \int v\,dt$

$s = \int (t^2 - \frac{1}{15}t^3 + \frac{15}{2})\,dt$

$= \frac{1}{3}t^3 - \frac{1}{15} \times \frac{1}{4}t^4 + \frac{15}{2}t + c$

$= \frac{1}{3}t^3 - \frac{1}{60}t^4 + \frac{15}{2}t + c$

The distance is measured from the start of the runway. So when $t = 0$, $s = 0$. Substitute these into the expression for s.

$0 = 0 - 0 + 0 + c$

$0 = c$

$s = \frac{1}{3}t^3 - \frac{1}{60}t^4 + \frac{15}{2}t$

To find the distance covered in the first 10 seconds substitute $t = 10$ into this expression for s.

$s = \frac{1}{3} \times 10^3 - \frac{1}{60} \times 10^4 + \frac{15}{2} \times 10$

$= \frac{1000}{3} - \frac{10\,000}{60} + \frac{150}{2}$

$= \frac{1000}{3} - \frac{1000}{6} + \frac{150}{2}$

$= \frac{2000}{6} - \frac{1000}{6} + \frac{450}{6}$

$= \frac{1450}{6} = \frac{725}{3}$

So $s = 241\frac{2}{3}$ m ◄ **Always include the appropriate unit.**

Note the exact answe

9.1 Indefinite Integration

Exercise

Technique

1 Use integration notation to describe each of the following statements:

a integrate $2x^2 - 3x + 1$ with respect to x
b integrate e^{2t} with respect to t
c integrate $\cos\theta$ with respect to θ.

2 Integrate each of the following with respect to x:

a 3 b $2x - 5$ c $x^2 + x - 7$
d $x(2 - 3x)$ e $(x - 3)(x + 5)$ f $(x + 1)^2$

$\boxed{2}$ e

3 Find:

a $\int \mathrm{d}x$ e $\int \frac{1}{t^2}\,\mathrm{d}t$ i $\int 100x^{1.2}\,\mathrm{d}x$
b $\int 10\,\mathrm{d}r$ f $\int x^{\frac{1}{3}}\,\mathrm{d}x$ j $\int \frac{1}{\sqrt{r}}\,\mathrm{d}r$
c $\int (t^3 + 1)\,\mathrm{d}t$ g $\int 5r^9\,\mathrm{d}r$ k $\int 3x^{-\frac{4}{3}}\,\mathrm{d}x$
d $\int \sqrt{x}\,\mathrm{d}x$ h $\int \frac{2}{t^3}\,\mathrm{d}t$

4 Find:

a $\int (x^2 - 3x + 2)\,\mathrm{d}x$ b $\int (2t - 5)^2\,\mathrm{d}t$ c $\int x(2 + x)\,\mathrm{d}x$
d $\int t(1 - t^2)^2\,\mathrm{d}t$ e $\int (x + 1)(x - 3)\,\mathrm{d}x$ f $\int 14x^{\frac{5}{2}}\,\mathrm{d}x$

5 Find:

a $\int \left(\frac{x^4 + 2}{x^3}\right)\mathrm{d}x$ b $\int \left(\frac{x + 3}{\sqrt{x}}\right)\mathrm{d}x$ c $\int \left(\frac{1}{t^3} + \frac{2}{t^2} + 3t\right)\mathrm{d}t$
d $\int (\sqrt{x} - 1)^2\,\mathrm{d}x$ e $\int \left(\frac{1 - 3x^2}{\sqrt{x}}\right)\mathrm{d}x$ f $\int (6 + x)\sqrt{x}\,\mathrm{d}x$

$\boxed{5}$ c, e

6 If $f'(x) = 8x$, and $f(1) = 4$, find $f(x)$.

7 If $\int f(x)\,\mathrm{d}x = g(x)$, what is the connection between $g(x)$ and $f(x)$?

Contextual

1 The velocity of a particle, at time t, is given by $v = t + 3$. Find an expression for the distance, s, travelled by the particle at time t, given that $s = 0$ when $t = 0$.

2 Copy and complete the following, where each '$\bullet$' represents a missing term:

- **a** $\int(2x^3 - 10x^2 - \bullet + \bullet)\,dx = \bullet - \frac{10}{3}x^3 - \frac{1}{2}x^2 + 3x + c$
- **b** $\int(\bullet - 15t)\,dt = \bullet\int(2 - 3t)\,dt = \bullet(\bullet - \bullet) + \bullet$
- **c** $\int dy = \bullet + \bullet$
- **d** $\int \bullet = \theta + c$
- **e** $\int(\bullet - \bullet + 1)\,dx = \frac{4}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + \bullet + \bullet$
- **f** $\int(\bullet - \bullet + t^5)\bullet = \frac{5}{2}t^2 - t^3 + \bullet + \bullet$
- **g** $\int(\bullet - 3)\bullet = u^2 - \bullet + \bullet$

3 If $g(x) = \int 6x(x+1)\,dx$:

- **a** write down $g'(x)$ and solve $g'(x) = 0$
- **b** find $g(x)$ given that $g(-1) = 5$.

4 The gradient of a curve at any point is given by $\frac{dy}{dx} = 4 - 2x$.

- **a** Explain why the curve has a maximum value when $x = 2$.
- **b** If the maximum value of the curve is 1 find the equation of the curve.

5 If $g'(x) = 1 - 4x^3$, and $(2, -13)$ is a point on the curve $g(x)$, find $g(x)$.

6 The gradient at any point (x, y) on a curve is given by $\frac{dy}{dx} = -\frac{4}{x^3}$. If the curve passes through the point $(2, 3\frac{1}{2})$, find the equation of the curve.

7 A train starts from rest at station A and $2\frac{1}{2}$ minutes later stops at station B. Its velocity, t seconds after starting, is given by $v = 0.6t - 0.004t^2\ \mathrm{m\,s^{-1}}$.

Model only applies during the time for t journey from A to B.

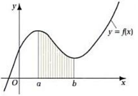
- **a** Find an expression for the distance travelled from A.
- **b** Find the distance between the two stations.

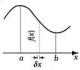
8 A particle leaves a fixed point O with a velocity of $20\ \mathrm{m\,s^{-1}}$. It travels in a straight line and its acceleration after t seconds is given by $a = (14 - 8t)\ \mathrm{m\,s^{-2}}$.

- **a** Find an expression for the velocity in terms of t.
- **b** Find an expression for the displacement of the particle after leaving O.

9.2 The Area Under a Graph

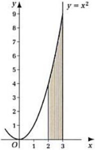
Integration is used for summation; usually of lengths, areas or volumes. The S-shaped symbol was devised by Leibniz because integration represents a sum. It was perceived as the sum of the areas of an infinite number of rectangles of height $f(x)$ and infinitesimally small width δx. So each rectangle would be thinner than a pencil line and its area would be $f(x)\,\delta x$.



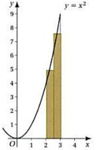


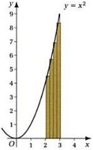
Height of pencil line is from the x-axis up to the graph.

We can find the area under the curve $y = x^2$, between $x = 2$ and $x = 3$, using Leibniz's idea that $\int_a^b f(x)\,dx$ is the sum of the areas of many thin rectangles.



Consider splitting the area into a series of thin rectangles.





When the area is split into two rectangles the width of each is 0.5. The height of the first rectangle is $f(2.25)$ and the height of the second rectangle as $f(2.75)$. The two distinct areas are then $0.5 \times f(2.25)$ and $0.5 \times f(2.75)$. Calculate these two values:

$$0.5 \times f(2.25) = 0.5 \times 5.0625 = 2.531\,25$$
$$0.5 \times f(2.75) = 0.5 \times 7.5625 = 3.781\,25$$

The total area under the curve is estimated to be

$$2.531\,25 + 3.781\,25 = 6.3125$$

Repeat the process, this time splitting the area into four rectangles. The width of each rectangle is then 0.25, and their heights are $f(2.125)$, $f(2.375)$, $f(2.625)$ and $f(2.875)$. Check that the total area is now 6.328 125. This process could be repeated many times. In each case the width of each rectangle gets much smaller. As the width becomes infinitesimally small (as $\delta x \to 0$), the heights become $f(x)$ for all values of x between $x = 2$ and $x = 3$. The total area of all the rectangles then becomes a better approximation to the exact area under the curve. The area can be calculated as

$$\sum_{x=2}^{3} f(x)\,\delta x$$

where this expression represents the sum of the areas of each rectangle with height $f(x)$ and width δx, between $x = 2$ and $x = 3$.

The limit as $\delta x \to 0$ of $\sum_{x=2}^{3} f(x)\,\delta x$ is $\int_2^3 f(x)\,dx$.

In general, adding the areas of all these rectangles gives the area under the graph between $x = a$ and $x = b$ as

area of the shaded region $= \int_a^b f(x)\,dx$ ◄ **Learn this result.**

We call 'b' the **upper limit** of the integral and 'a' the **lower limit** of the integral. An integral with limits is a **definite integral**.
The shaded area under the curve $y = x^2$ is given by $\int_2^3 x^2\,dx$.

$$\int_2^3 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_2^3$$
◄ **Square brackets are used to separate the limits from the integrated expression.**
$$= \left(\tfrac{27}{3}\right) - \left(\tfrac{8}{3}\right)$$
$$= \tfrac{19}{3} = 6\tfrac{1}{3}$$

So the area under the curve is $6\frac{1}{3}$ square units. Compare this to the result for four rectangles.

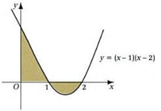
Graphic
calculat
support
pack

You could investigate
the sum of the rectan
areas further using th
program option on a
graphical calculator.

The constant term do
not have to be writter
down for a definite
integral.

Putting $x = 3$ and $x =$
gives the area under t
graph between $x = 0$
$x = 3$, and $x = 0$ and
respectively. Subtrac
the second from the f
gives the area under t
graph between $x = 2$
$x = 3$.

Example 1



The graph shows the curve $y = (x - 1)(x - 2)$.

a Find $\int_0^1 (x-1)(x-2)\,dx$ and $\int_1^2 (x-1)(x-2)\,dx$.
Comment on your answer.

b What would you expect $\int_0^2 (x-1)(x-2)\,dx$ to equal?

c Find the area of the shaded region.

Solution

a To use integration first express y as a polynomial in x.

$$y = (x-1)(x-2) = x^2 - 3x + 2$$

$$\int_0^1 (x-1)(x-2)\,dx = \int_0^1 (x^2 - 3x + 2)\,dx$$

Integrating each term separately.

$$= \left[\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + 2x\right]_0^1$$
$$= \left(\tfrac{1}{3} - \tfrac{3}{2} + 2\right) - (0 - 0 + 0)$$
$$= \tfrac{2}{6} - \tfrac{9}{6} + \tfrac{12}{6}$$
$$= \tfrac{5}{6}$$

The area between the curve $y = (x-1)(x-2)$ and the x-axis between $x = 0$ and $x = 1$ is $\frac{5}{6}$ square units.

$$\int_1^2 (x-1)(x-2)\,dx = \left[\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + 2x\right]_1^2$$

From the first part of the question.

$$= \left(\tfrac{1}{3} \times 8 - \tfrac{3}{2} \times 4 + 2 \times 2\right) - \left(\tfrac{1}{3} - \tfrac{3}{2} + 2\right)$$
$$= \tfrac{16}{6} - \tfrac{36}{6} + \tfrac{24}{6} - \tfrac{2}{6} + \tfrac{9}{6} - \tfrac{12}{6}$$
$$= -\tfrac{1}{6}$$

The value of the integral between $x = 1$ and $x = 2$ is $-\frac{1}{6}$. This is interpreted as an area of $\frac{1}{6}$ square units below the x-axis.

The value of the integral between $x = 1$ and $x = 2$ is negative. This is because the region bounded by the curve, $x = 1$, $x = 2$ and the x-axis lies below the x-axis.

b Since integration is a summation process we might expect that

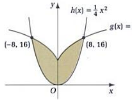
$$\int_0^2 (x-1)(x-2)\,dx = \int_0^1 (x-1)(x-2)\,dx + \int_1^2 (x-1)(x-2)\,dx$$
$$= \tfrac{5}{6} + \left(-\tfrac{1}{6}\right) = \tfrac{4}{6} = \tfrac{2}{3}$$

c Looking at the sketch of the function, we can avoid adding the areas algebraically. The graph shows that part of the curve is below the x-axis, and for this region the integral is negative. Adding the integrals, as in **b**, would give too small a result for the area. Therefore,

total shaded area $= \frac{5}{6} + \frac{1}{6} = \frac{6}{6} = 1$ square unit.

A sketch of the curve will always give you a indication of whether the curve crosses the x-axis. If it does, you must calculate the ar of the regions above a below the axis separately, and then a them together to find total area.

Example 2

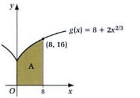


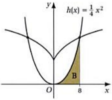
Find the area of shaded region enclosed between $g(x) = 8 + 2x^{\frac{2}{3}}$ and $h(x) = \frac{1}{4}x^2$.

Solution

The area is symmetrical, with the y-axis as the line of symmetry. So we can find the area of the whole shaded region by doubling the area of the part of the region lying in the first quadrant. Call the area under the curve $g(x)$ area A, and the area under the curve $h(x)$ area B.

The first quadrant is t part of the curve whe and y coordinates are both positive; above t x-axis and to the righ the y-axis.





On the diagram, notic the coordinates of th point where $g(x)$ and $h(x)$ cross. The x-coordinate is 8, so v require the area betw $x = 0$ and $x = 8$.

$$\text{area A} = \int_0^8 (8 + 2x^{\frac{2}{3}})\,dx$$
$$= \left[8x + \frac{2}{\frac{5}{3}}x^{\frac{5}{3}}\right]_0^8$$
$$= \left[8x + \tfrac{6}{5}x^{\frac{5}{3}}\right]_0^8$$
$$= (64 + \tfrac{6}{5} \times 32) - (0 + 0) = \tfrac{512}{5}$$

$$\text{area B} = \int_0^8 \tfrac{1}{4}x^2\,dx = \tfrac{1}{4}\int_0^8 x^2\,dx$$
$$= \tfrac{1}{4}\left[\tfrac{1}{3}x^3\right]_0^8$$
$$= \tfrac{1}{4}\left[(\tfrac{1}{3} \times 512) - 0\right]$$
$$= \tfrac{512}{12} = \tfrac{128}{3}$$

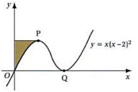
Recall that $\int kf(x)\,dx = k\int f(x)\,d$

So the area of the shaded region $= 2 \times (\text{area A} - \text{area B})$

$$= 2 \times \left(\tfrac{512}{5} - \tfrac{128}{3}\right) = \tfrac{1792}{15}$$

Area of the shaded region = 119.47 square units (2 d.p.).

Example 3



The graph shows a sketch of the curve $y = x(x-2)^2$.

a Find the coordinates of P.
b Find the shaded area.

Solution

a
$$\begin{aligned} y &= x(x-2)^2 \\ &= x(x^2 - 4x + 4) \\ &= x^3 - 4x^2 + 4x \end{aligned}$$
$$\Rightarrow \tfrac{dy}{dx} = 3x^2 - 8x + 4$$

P is a local maximum point, so $\frac{dy}{dx} = 0$ at P. That is, $3x^2 - 8x + 4 = 0$.

This is a quadratic equation. The coefficient of x^2 is larger than 1, so factorise and solve it using PAFF:

P: $3 \times 4 = 12$
A: -8
F: $-2, -6$

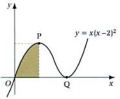
$$\begin{aligned} 3x^2 - 8x + 4 &= 3x^2 - 2x - 6x + 4 \\ &= x(3x-2) - 2(3x-2) \\ &= (3x-2)(x-2) \end{aligned}$$

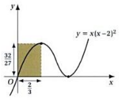
When $3x^2 - 8x + 4 = 0$, $(3x-2)(x-2) = 0$.
So $x = \frac{2}{3}$ or $x = 2$.

Substituting each of these coordinates into the original equation, we find that the coordinates of P are $\left(\frac{2}{3}, \frac{32}{27}\right)$ and the coordinates of Q are $(2, 0)$.

Check these against the sketch of the graph.

b The area of the shaded region can be found by subtracting the area of the region below the graph between $x = 0$ and $x = \frac{2}{3}$ from the area of the rectangle with corners $(0, 0)$, $(0, \frac{32}{27})$, P $(\frac{2}{3}, \frac{32}{27})$ and $(\frac{2}{3}, 0)$.





The area of the rectangle is $\frac{2}{3} \times \frac{32}{27} = \frac{64}{81}$

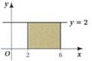
The area under the graph $y = x(x-2)^2$ between $x = 0$ and $x = \frac{2}{3}$ is given by

$$\begin{aligned}
\text{area} &= \int_0^{\frac{2}{3}} (x^3 - 4x^2 + 4x)\,dx \\
&= \left[\tfrac{1}{4}x^4 - \tfrac{4}{3}x^3 + 2x^2\right]_0^{\frac{2}{3}} \\
&= \left(\tfrac{1}{4} \times \tfrac{16}{81} - \tfrac{4}{3} \times \tfrac{8}{27} + 2 \times \tfrac{4}{9}\right) - (0 - 0 + 0) \\
&= \tfrac{4}{81} - \tfrac{32}{81} + \tfrac{72}{81} \\
&= \tfrac{44}{81}
\end{aligned}$$

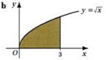
So the area of the original shaded region $= \frac{64}{81} - \frac{44}{81} = \frac{20}{81}$ square units.

4 Use integration to find the area of the shaded region in each of the following:

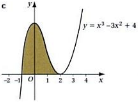
a



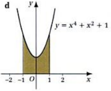
b



c



d



5 Evaluate the following definite integrals:

a $\int_1^2 (5x^4 - 6x^3 + x + 1)\,dx$ **b** $\int_{-2}^{3} (x^2 - x - 6)\,dx$

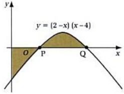
c $\int_{-3}^{1} (10 - 2x + 6x^2)\,dx$ **d** $\int_{-2}^{1} (1 - 10x^4)\,dx$

6 The graph shows a sketch of the curve $y = (2 - x)(x - 4)$.

a Write down the coordinates of P and Q.

b Explain why it is necessary to evaluate two separate integrals in order to calculate the area of the shaded region.

c Calculate the area of the shaded region.

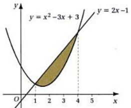


7 This diagram shows the curve $y = x^2 - 3x + 3$ and the line $y = 2x - 1$.

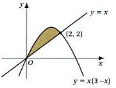
a Find the value of $\int_1^4 (x^2 - 3x + 3)\,dx$.

b Find the value of $\int_1^4 (2x - 1)\,dx$.

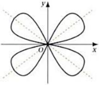
c Hence find the area of the shaded region.



8 **a** Find the area of this shaded region.



b Now find the area of this pattern.



9 Evaluate the following integrals:

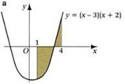
a $\int_1^4 \frac{2}{\sqrt{x}}\,dx$ **b** $\int_0^{16} (\sqrt{x} - 1)\,dx$

c $\int_2^6 \frac{4}{x^3}\,dx$ **d** $\int_1^4 5x^{\frac{3}{2}}\,dx$

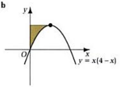
Contextual

1 Find the area of the shaded region in each of the following:

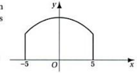
a



b



2 The diagram shows the vertical section through a tunnel 14 m long. The roof is an arc modelled by the equation $y = 6 - 0.08x^2 - 0.0006x^4$.



a Find the area of the cross-section.

b Find the volume of the tunnel.

9.3 The Area Between the Vertical Axis and a Curve

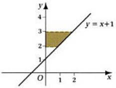
Exercise

Technique

1 The graph shows the line $y = x + 1$.

a Find x in terms of y.

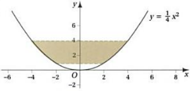
b Use integration to find the sum of an infinite number of horizontal rectangles that describe the shaded area.



Check your answer for **b** by finding the shaded area as the area of a trapezium.

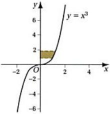
2

2

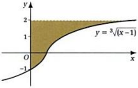


The graph shows the curve $y = \frac{1}{4}x^2$. Express x in terms of y and then calculate the area of the shaded region.

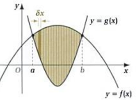
3 The graph shows the curve $y = x^3$. Express x in terms of y. Hence calculate the area of the shaded region, correct to two decimal places.

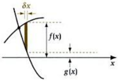


4 The curve shows the graph of $y = \sqrt[3]{(x-1)}$. Express x in terms of y. Hence calculate the exact area of the shaded region.



9.4 The Area Between Two Curves





The shaded region is enclosed between the graphs of $y = f(x)$ and $y = g(x)$. Divide the area into vertical strips. Notice that when $f(x) > g(x)$ the length of each strip is $f(x) - g(x)$. This is the case for all values of x between $x = a$ and $x = b$. Adding together the areas of all the thin rectangular strips gives the area of the shaded region.

The area of each strip is $(f(x) - g(x)) \times \delta x$.

The area between the two curves is $\sum_{x=a}^{b} (f(x) - g(x))\,\delta x$.

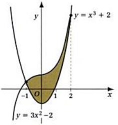
As $\delta x \to 0$ and the width of each strip becomes extremely small, the area can be calculated by integration.

The area between the two curves is $\int_a^b (f(x) - g(x))\,\mathrm{d}x$.

This method is very useful if one of the curves lies below the x-axis.

Example 1

Find the area enclosed between the curves $y = x^3 + 2$ and $y = 3x^2 - 2$.



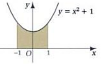
Solution

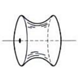
Since $y = x^3 + 2$ is greater than $y = 3x^2 - 2$ between $x = -1$ and $x = 2$, use

$$\text{area} = \int_a^b (f(x) - g(x))\,\mathrm{d}x$$

where $f(x) = x^3 + 2$ and $g(x) = 3x^2 - 2$.

Solution





The solid formed by the rotation

$$\text{volume} = \int_a^b \pi y^2 \, dx$$

Recall that the dx indicates that the integration is carried out with respect to x.

First find an expression for y^2 in terms of x.

$$y = x^2 + 1$$

$$\Rightarrow y^2 = (x^2 + 1)^2 = x^4 + 2x^2 + 1$$

$$\begin{aligned}\text{So volume of revolution} &= \int_{-1}^{1} \pi(x^4 + 2x^2 + 1)\, dx \\ &= \pi \int_{-1}^{1} (x^4 + 2x^2 + 1)\, dx \\ &= \pi\left[\tfrac{1}{5}x^5 + \tfrac{2}{3}x^3 + x\right]_{-1}^{1} \\ &= \pi\{(\tfrac{1}{5} + \tfrac{2}{3} + 1) - (-\tfrac{1}{5} - \tfrac{2}{3} - 1)\} \\ &= \pi(\tfrac{28}{15} + \tfrac{28}{15}) = \tfrac{56\pi}{15}\end{aligned}$$

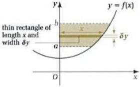
The limits, $a = -1$ and $b = 1$, are read from the sketch.

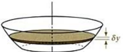
Notice that this exact result can be approximated to 11.7 cubic units (3 s.f.).

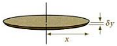
Check this result using a graphical calculator.

Rotation about the *y*-axis

Find the volume generated when the shaded area bounded by $y = f(x)$ and the y-axis, from $y = a$ to $y = b$, is rotated about the y-axis.
The width of the approximate rectangle, δy, is a small measurement along the y-axis.
Rotating the shaded area about the y-axis forms a solid.
The rectangle becomes a thin horizontal disc, of depth δy and radius x.







The volume of the thin disc is $\pi x^2 \delta y$.

The volume of the solid of revolution is found by adding together the volumes of all the discs.

Volume of solid revolution $= \sum_{y=a}^{b} \pi x^2 \, \delta y$, where δy is infinitesimally small.

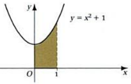
We use integration to sum the volumes of the discs.

volume of revolution about the y-axis $= \int_a^b \pi x^2 \, dy$

Compare this result with the formula for rotation about the x-axis. In particular notice the $y^2 dx$ and $x^2 dy$ terms.

Recall that dy means integrate 'with respec to' y.

Example 2



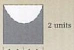
Find the volume of the solid generated by rotating the shaded region about the y-axis.

Solution

Make a sketch of the solid.

Given $y = x^2 + 1$, wh $x = 1$, $y = 1^2 + 1 = 2$ the height of the soli 2 units.

cross-section

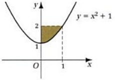


It is a cylinder with a bowl shape removed, so we can find its volume by subtracting the volume of the bowl shape from the volume of the cylinder.

$$\begin{aligned}\text{volume of cylinder} &= \pi r^2 h \\ &= \pi \times 1^2 \times 2 \\ &= 2\pi \text{ cubic units}\end{aligned}$$

The coordinates of the curve at each edge of the shaded region are $(0, 1)$ and $(1, 2)$, so the bowl shaped solid is obtained by rotating the region between the y-axis and the curve between $y = 1$ and $y = 2$ through $360°$.

To use the equation that we know for volumes of revolution around the y-axis, we need to find the equation of the curve in terms of x^2.



$$y = x^2 + 1 \quad \Rightarrow \quad y - 1 = x^2$$

$$\begin{aligned}
\text{volume} &= \int_1^2 \pi x^2 \, dy \\
&= \int_1^2 \pi(y - 1) \, dy \\
&= \pi\left[\tfrac{1}{2}y^2 - y\right]_1^2 \\
&= \pi[(\tfrac{1}{2} \times 4 - 2) - (\tfrac{1}{2} \times 1 - 1)] \\
&= \tfrac{1}{2}\pi \text{ cubic units}
\end{aligned}$$

So the required volume $= 2\pi - \frac{1}{2}\pi = \frac{3}{2}\pi$ cubic units.

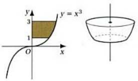
Volume = 4.71 cubic units (3 s.f.)

Example 3

Find the volume generated when the region bounded by the curve $y = x^3$, the y-axis, and the lines $y = 1$ and $y = 3$ is rotated through four right angles about the y-axis.

Solution

First, sketch the curve and the solid of revolution formed by the rotation about the y-axis.



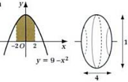
$$\begin{aligned}
\text{volume of revolution} &= \int_a^b \pi x^2 \, dy \\
&= \pi \int_1^3 x^2 \, dy \\
&= \pi \int_1^3 y^{\frac{2}{3}} \, dy \\
&= \pi \left[\frac{1}{\frac{5}{3}} y^{\frac{5}{3}}\right]_1^3 \\
&= \pi \left[\tfrac{3}{5} y^{\frac{5}{3}}\right]_1^3 \\
&= \pi[(\tfrac{3}{5} \times 3^{\frac{5}{3}}) - (\tfrac{3}{5} \times 1)] \\
&= \frac{3\pi}{5}(3^{\frac{5}{3}} - 1) \\
&= 9.88 \text{ cubic units (3 s.f.)}
\end{aligned}$$

$y = x^3 \Rightarrow y^{\frac{1}{3}} = x$

Then $x^2 = (y^{\frac{1}{3}})^2 = y^{\frac{2}{3}}$

9.5 Volumes of Revolution
Exercise
Technique

1 Find the volume of each of the solids formed by rotating the following areas about the x-axis, showing clearly the necessary integration:

a


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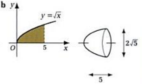
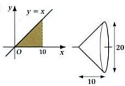
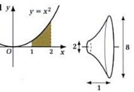
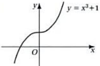
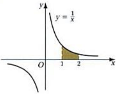


c


d


2


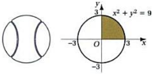
Sketch the solid formed when the area under the curve $y = x^3 + 1$, from $x = -1$ to $x = 2$, is rotated through four right angles about the x-axis. Calculate the volume of this solid.

3


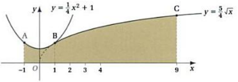
Find the volume of the solid formed when the shaded area is rotated about the x-axis.

Contextual

1 A ball has a diameter of 6 cm. Its volume can be calculated using integration. The equation of a circle of radius 3 is $x^2 + y^2 = 9$. Find the volume of the hemisphere formed when the area under the curve between $x = 0$ and $x = 3$ is rotated through one complete turn about the x-axis. Hence find the volume of the ball. Check this result using the formula for volume of a sphere.

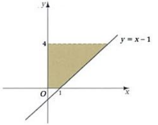


2



A small vase is formed when the shaded region is rotated through one complete turn about the x-axis. The arc AB is the part of the curve $y = \frac{1}{4}x^2 + 1$ between $x = -1$ and $x = 1$. The arc BC is the part of the curve $y = \frac{5}{4}\sqrt{x}$ between $x = 1$ and $x = 9$. Find the volume of the vase.

3



A funnel is formed when the shaded region, bounded by the line $y = x - 1$, the y-axis, and the lines $y = 0$ and $y = 4$, is rotated through one complete turn about the y-axis. Find the volume of the funnel.

c Recall that $\frac{d}{dx}(\sin(ax+b)) = a\cos(ax+b)$
This means $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

d $\int \sin^2 x\,dx$ is not one of the standard integrals. Use the double angle formula from Chapter 3 to rewrite $\sin^2 x$ in terms of a multiple angle.

$$\cos 2x = 1 - 2\sin^2 x$$

$$\Rightarrow\ 2\sin^2 x = 1 - \cos 2x$$

$$\Rightarrow\ \sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x$$

Now $\int \sin^2 x\,dx = \int (\tfrac{1}{2} - \tfrac{1}{2}\cos 2x)\,dx$

$$= \tfrac{1}{2}\int (1 - \cos 2x)\,dx$$

$$= \tfrac{1}{2}(x - \tfrac{1}{2}\sin 2x) + c$$

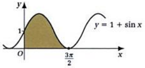
$$= \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + c$$

Since we could just write a new constant of integration equal to $\frac{1}{2}c$, there is no need to include c in the part of the integral multiplied by $\frac{1}{2}$.

Example 2

Find the area enclosed by the x-axis, y-axis and the curve $y = 1 + \sin x$.

Solution



Sketch the curve to establish the limits.

Notice the translation of the sine curve.

$$\text{area} = \int_0^{3\pi/2} (1 + \sin x)\,dx$$

$$= [x - \cos x]_0^{3\pi/2}$$

$$= \left[\frac{3\pi}{2} - \cos\left(\frac{3\pi}{2}\right)\right] - (0 - \cos 0)$$

$$= \frac{3\pi}{2} - 0 - 0 + 1$$

$$= \frac{3\pi}{2} + 1$$

Recall the special angles in radian form; $\cos(\frac{3\pi}{2}) = 0$.

Example 3

Evaluate $\int_0^{\pi/12}(\cos 4x - 6\sin 2x)\,dx$.

Solution

$$\begin{aligned}\int_0^{\pi/12}(\cos 4x - 6\sin 2x)\,dx &= \left[\tfrac{1}{4}\sin 4x - 6\times\tfrac{1}{2}(-\cos 2x)\right]_0^{\pi/12} \\ &= \left[\tfrac{1}{4}\sin 4x + 3\cos 2x\right]_0^{\pi/12} \\ &= \left[\frac{1}{4}\sin\left(\frac{\pi}{3}\right) + 3\cos\left(\frac{\pi}{6}\right)\right] \\ &\quad - \left(\frac{1}{4}\sin 0 + 3\cos 0\right) \\ &= \left(\frac{1}{4}\times\frac{\sqrt{3}}{2}\right) + \left(3\times\frac{\sqrt{3}}{2}\right) - 0 - 3 \\ &= \frac{\sqrt{3}}{8} + \frac{3\sqrt{3}}{2} - 3 \\ &= \frac{13\sqrt{3}}{8} - 3\end{aligned}$$

Take care with the si[gn] changes.

Special angles:

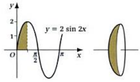
$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Add the fractions.

Example 4

Find the volume of the solid formed when the shaded region is rotated through 360° about the x-axis.



Solution

Since the solid has been formed by a rotation about the x-axis, use

$$\text{volume} = \int_a^b \pi y^2\,dx$$

We know that $y = 2\sin 2x$, so $y^2 = 4\sin^2 2x$.
From the graph, the limits are $a = 0$ and $b = \frac{1}{4}\pi$.

$$\begin{aligned}\text{volume} &= \int_0^{\pi/4}\pi(4\sin^2 2x)\,dx \\ &= 4\pi\int_0^{\pi/4}\sin^2 2x\,dx\end{aligned}$$

Taking the constant multipliers out of the integrand.

The integrand contains $\sin^2 2x$, which we can't integrate directly. We must use the multiple angle formula to write it in terms of double angles.

$$\cos 2A = 1 - 2\sin^2 A \Rightarrow \sin^2 A = \tfrac{1}{2} - \tfrac{1}{2}\cos 2A$$

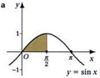
Putting $A = 2x$ gives $\sin^2 2x = \frac{1}{2} - \frac{1}{2}\cos 4x$

$$\begin{aligned}
\text{volume} &= 4\pi \int_0^{\pi/4} \sin^2 2x \, dx \\
&= 4\pi \int_0^{\pi/4} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 4x\right) dx \\
&= 2\pi \int_0^{\pi/4} (1 - \cos 4x) \, dx \\
&= 2\pi \left[x - \tfrac{1}{4}\sin 4x\right]_0^{\pi/4} \\
&= 2\pi \left[\left(\tfrac{\pi}{4} - \tfrac{1}{4}\sin \pi\right) - \left(0 - \tfrac{1}{4}\sin 0\right)\right] \\
&= 2\pi \times \left(\tfrac{\pi}{4} - 0 - 0 + 0\right) \\
&= \tfrac{1}{2}\pi^2 \text{ cubic units}
\end{aligned}$$

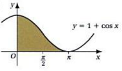
Take a factor of $\frac{1}{2}$ outside the integral.

7 Find the areas of the following shaded regions:

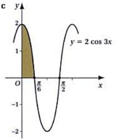
a



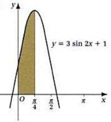
b



c



d



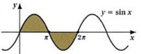
8 Find:

a $\int \sin^2 x \, dx$

b $\int \cos^2 x \, dx$

Contextual

1



The graph shows $y = \sin x$.

a Explain why $\int_0^{2\pi} \sin x \, dx = 0$.

b What area is represented by $\int_0^{\pi/2} \sin x \, dx$? Evaluate this integral.

c Use your answer to **b** to calculate the area of the shaded region.

2 A particle moves in a straight line. Its velocity at time t is given by $v = 12\cos 3t$. Find an expression for the displacement, s, after t seconds, if $s = 0$ when $t = 0$.

Consolidation

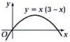
Exercise A

1 Find:

a $\int(2x-3)^2\,dx$

b the equation for a curve, $y = f(x)$, for which $\frac{dy}{dx} = 6x - 1$ and which passes through the point $(1, 9)$.

2 The sketch shows the graph of $y = x(3-x)$.

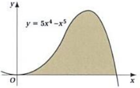


a Find $\int x(3-x)\,dx$.

b Hence calculate the area between the curve and the x-axis.

(SMP)

3 The sketch shows a graph of the curve $y = 5x^4 - x^5$.



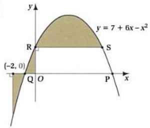
a Find $\frac{dy}{dx}$ and calculate the coordinates of the stationary points.

b Calculate the area of the shaded region enclosed between the curve and the x-axis.

c Evaluate $\int_0^6 x^4(5-x)\,dx$, and comment on your result.

(MEI)

4



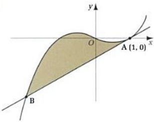
The diagram shows part of the curve $y = 7 + 6x - x^2$. Find:

a the coordinates of the points P, Q, R and S.

b the area of each shaded region.

(UCLES)

5



The sketch shows the graph of $y = x^3 - x$ together with the tangent to the curve at the point A $(1, 0)$.

a Use differentiation to find the equation of the tangent to the curve at A and verify that the point B where the tangent cuts the curve again has coordinates $(-2, -6)$.

b Use integration to find the area of the region bounded by the curve and the tangent (shaded in the diagram), giving your answer as a fraction in its lowest terms.

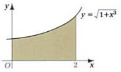
(UCLES)

6 **a** Find the area of the region enclosed by the curve $y = \frac{12}{x^2}$, the x-axis and the lines $x = 1$ and $x = 3$.

b The area of the region enclosed by $y = \frac{12}{x^2}$, the x-axis and the lines $x = 2$ and $x = a$, where $a > 2$, is 3.6 units squared. Find the value of a.

(UCLES)

7



The diagram shows part of the curve $y = \sqrt{1 + x^3}$. Calculate the volume formed when the shaded region is rotated through $360°$ about the x-axis.

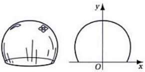
8 **a** Find $\int (2\cos x - 3\sin x)\,dx$.

b Find $\int (1 - 6\sin 3x)\,dx$.

c Evaluate $\int_0^{\pi/2} 2\cos 2t\,dt$.

d Sketch the curve $y = 1 + \cos x$. Find the area enclosed between the axes and curve from $x = 0$ to $x = \pi$.

9



The shape of a glass paperweight is that of the solid formed by rotating about the y-axis the part of the curve $3x^2 + 2y^2 - 12y = 32$ for which $y \geq 0$ (in centimetres).

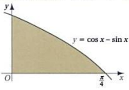
a Verify that the paperweight is 8 cm high.

b Calculate the volume of the paperweight. You may leave π in your answer.

(SMP)

10 **a** Show that $(\cos x - \sin x)^2 = 1 - \sin 2x$.

b



The diagram shows part of the curve $y = \cos x - \sin x$. Find the volume generated when the shaded region is rotated through a complete revolution about the x-axis.

(UCLES)

Exercise B

1 **a** Find $\int \left(\sqrt{x} + \frac{12}{x^2}\right) \mathrm{d}x$.

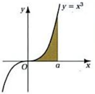
b Hence evaluate $\int_4^9 \left(\sqrt{x} + \frac{12}{x^2}\right) \mathrm{d}x$.

(NEAB)

2 **a** A curve satisfies $\frac{\mathrm{d}y}{\mathrm{d}x} = (3x - 1)^2$. Given that the curve passes through $(1, 4)$, find its equation.

b A particle moves in a straight line so that t seconds after leaving a fixed point A, its velocity v ($\mathrm{m\,s^{-1}}$) is given by $v = t^2 - 4t + 3$. Find the displacement, s metres, for the particle at t seconds, if $s = 0$ when $t = 0$.

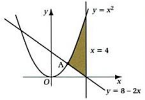
3 The sketch shows the curve $y = x^3$. Find a if the area of the shaded region is 1 square unit.



4 The diagram shows the curve $y = x^2$, and the lines $x = 4$ and $y = 8 - 2x$.

a Find the coordinates of A.

b Find the area of the shaded region.



5 **a** On the same graph, sketch the line with equation $y = x + 1$ and the curve with equation $y = 5x - x^2 + 6$. Values of x should be taken from $x = -2$ to $x = +8$. Shade in the region between the line and the curve.

b Calculate the points of intersection of the line $y = x + 1$ and the curve $y = 5x - x^2 + 6$.

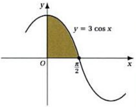
c Use integration to calculate the area of the region that you shaded in **a**.

(MEI)

6 A cup has the shape made by rotating the graph of $y = 3x^2$, for $0 \leq x \leq 1$, through four right angles about the y-axis. Find the volume of the cup, giving your answer in terms of π.

(UCLES)

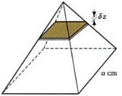
7 The graph shows part of the curve $y = 3\cos x$. The shaded region is rotated about the x-axis through four right angles. Calculate the volume formed.

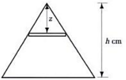


Applications and Activities

1 *Volume of a pyramid*

It has been documented that the Greek Democritus (460–361 BC) may have calculated the volume of a pyramid by considering it as an infinite number of thin cross-sections. Let A(z) be the area of the horizontal cross-section z cm from the top, and δz be the thickness of the cross-section.





vertical section

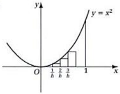
a Find A(z) in terms of z, a and h.

b Investigate how to calculate $\sum_{z=0}^{h} \mathrm{A}(z)\,\delta z$.

c Find the volume of the pyramid.

2 *Area under a curve*

The area under the curve $y = x^2$ between $x = 0$ and $x = 1$ is divided up into h rectangles of equal width.



a Using $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, show that the area of all the rectangles can be written $\frac{1}{6h^2}(h-1)(2h-1)$.

b Investigate the total area under the curve $y = x^2$ between $x = 0$ and $x = 1$ for different values of h.

c Explain why $\int_0^1 x^2\,dx > \frac{1}{6h^2}(h-1)(2h-1)$.

d By writing $\frac{1}{6h^2}(h-1)(2h-1)$ in a different form, deduce its value as $h \to \infty$. Comment on your answer.

Graphical calculator support pack

This can be investigated using a graphical calculator or spreadsheet.

10 Trigonometry II

What you need to know

- The compound and double angle formulas.
- How to convert angle measurements between degrees and radians.
- The trigonometric ratios for the 'special angles'.

Review

1 Write down the compound angle expansions for the following:

a $\sin(A+B)$
b $\cos(A-B)$
c $\tan 2A$
d $\cos(x+60^\circ)$
e $\sin(x-30^\circ)$
f $\tan(x-45^\circ)$

2 **a** Convert the following angles into radians, leaving the answers in terms of π:

i 45°
ii 120°
iii 315°
iv 220°
v 400°
vi -40°

b Convert the following angles into degrees:

i $\frac{\pi}{6}$
ii $\frac{\pi}{2}$
iii $\frac{5\pi}{4}$
iv $\frac{\pi}{20}$
v $\frac{11\pi}{9}$
vi $-\frac{7\pi}{10}$

3 Without using a calculator, write down the exact value of the following:

a $\sin 60^\circ$
b $\tan\frac{\pi}{6}$
c $\cos 90^\circ$
d $\sin\frac{2\pi}{3}$
e $\tan\pi$
f $\cos 120^\circ$

10.1 The Factor Formulas

In Chapter 3 we established the concept of an identity as an equation that is true for all values of the variable. One set of trigonometric identities is known as the **factor formulas.** They convert expressions like $\sin A + \sin B$ into a product; similar to factorising the expression. Another common name for these identities is **sum and product formulas.**

$$\sin C + \sin D \equiv 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \quad [1]$$

$$\sin C - \sin D \equiv 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \quad [2]$$

$$\cos C + \cos D \equiv 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \quad [3]$$

$$\cos C - \cos D \equiv -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \quad [4]$$

These four identities have a straightforward derivation from the compound angle formulas encountered in Chapter 3. To derive identity [1], recall the formulas:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

Adding these two equations,

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$

Now let $C = A+B$ and $D = A-B$.
The LHS becomes $\sin C + \sin D$, and we have
$C + D = (A+B) + (A-B) = 2A$ and
$C - D = (A+B) - (A-B) = A + B - A + B = 2B$

So $A = \frac{1}{2}(C+D)$ and $B = \frac{1}{2}(C-D)$

The RHS of the equation becomes $2\sin[\frac{1}{2}(C+D)]\cos[\frac{1}{2}(C-D)]$

$$\text{So } \sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

Use the same technique to derive identity [2].

Subtract the result $\sin(A-B)$ from $\sin(A+B)$.

To derive identities [3] and [4] we need to use the compound angle formulas for cosine.

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

Identity [3] can be derived by adding these results.

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$$

Now let $C = (A+B)$ and $D = (A-B)$ as before. We find that:

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

Derive identity [4], by subtracting the cosine results instead of adding them.

Note carefully the symmetries and patterns in the four factor formulas. They can often be more easily remembered if you say the words as well as write the symbols, for example:

'sine plus sine equals twice the sine of half the sum,

cos of half the difference'.

Alternatively, 'sine plus sine equals twice the sine the semi sum, cos the semi difference'. Write down a phrase you will remember for the other formulas, taking care to spot the negative signs.

The factor formulas provide a powerful mathematical tool for dealing with trigonometric functions. They can be used to solve equations, simplify expressions and prove more identities.

Example 1

Find $\sin 105^\circ - \sin 15^\circ$, without using a calculator.

Solution

$$\begin{aligned}\sin 105^\circ - \sin 15^\circ &= 2\cos\tfrac{1}{2}(105^\circ + 15^\circ)\sin\tfrac{1}{2}(105^\circ - 15^\circ)\\ &= 2\cos 60^\circ \sin 45^\circ\\ &= 2 \times \tfrac{1}{2} \times \tfrac{\sqrt{2}}{2} = \tfrac{\sqrt{2}}{2}\end{aligned}$$

Using the factor formula for the difference of two sines and recalling special angles.

Example 2

Prove the following identities:

a $\cos 2\theta + \cos 3\theta + \cos 4\theta \equiv \cos 3\theta(1 + 2\cos\theta)$

b $\dfrac{\sin 3\theta + \sin\theta}{\cos 3\theta + \cos\theta} \equiv \tan 2\theta$

Solution

a Notice that the LHS has three terms, one of which is $\cos 3\theta$. This also appears on the RHS so it would be sensible to apply one of the factor formulas on the other two terms.

$$\begin{aligned}\text{LHS} &= \cos 2\theta + \cos 3\theta + \cos 4\theta \\ &= \cos 4\theta + \cos 2\theta + \cos 3\theta \\ &= 2\cos 3\theta \cos\theta + \cos 3\theta \\ &= \cos 3\theta(2\cos\theta + 1) \\ &= \text{RHS}\end{aligned}$$

$$\cos 2\theta + \cos 3\theta + \cos 4\theta \equiv \cos 3\theta(1 + 2\cos\theta)$$

Reorder so that the factor formula for the sum of two cosines is straightforward to apply.

Factorising.

b

$$\begin{aligned}\text{LHS} &= \frac{\sin 3\theta + \sin\theta}{\cos 3\theta + \cos\theta} \\ &= \frac{2\sin 2\theta\cos\theta}{\cos 3\theta + \cos\theta} \\ &= \frac{2\sin 2\theta\cos\theta}{2\cos 2\theta\cos\theta} \\ &= \frac{\sin 2\theta}{\cos 2\theta} \\ &= \tan 2\theta = \text{RHS}\end{aligned}$$

$$\frac{\sin 3\theta + \sin\theta}{\cos 3\theta + \cos\theta} \equiv \tan 2\theta$$

Apply the factor formulas separately to the numerator and denominator.

Example 3

Use the factor formulas to express the following as a sum or difference of two trigonometrical functions:

a $2\sin 2A\cos A$ **b** $-2\sin 4A\sin 2A$

Solution

a Notice that $2\sin 2A\cos A$ is a product of two trigonometric terms. Since it contains one sine and one cosine term, it must be the result of summing two sine terms. Compare it with the factor formulas.

If $\sin x + \sin y = 2\sin 2A\cos A$, then

$$\begin{aligned} x + y &= 4A \\ \text{and} \quad x - y &= 2A\end{aligned}$$

'Half the sum' must be 2A and 'half the difference' must be A.

The solutions of these simultaneous equations are $x = 3A$ and $y = A$.

So $2\sin 2A\cos A = \sin 3A + \sin A$

b $-2\sin 4A\sin 2A$ contains two sine terms and a negative sign. The factor formulas suggest that this is a difference of two cosines.

If $-2\sin 4A \sin 2A = \cos x - \cos y$, then

$$x + y = 8A$$
and $$x - y = 4A$$

These simultaneous equations have solutions $x = 6A$ and $y = 2A$.

So $-2\sin 4A \sin 2A = \cos 6A - \cos 2A$.

Example 4

Without using a calculator, evaluate $2\cos 75^\circ \cos 15^\circ$.

Solution

Notice the two cosine terms. The factor formulas suggest that this is created from the sum of two cosines.

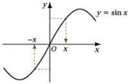
If $2\cos 75^\circ \cos 15^\circ = \cos x + \cos y$, then ◀ Identity 3.

$$x + y = 150^\circ$$
and $$x - y = 30^\circ$$

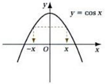
These simultaneous equations have solutions $x = 90^\circ$ and $y = 60^\circ$.

So $2\cos 75^\circ \cos 15^\circ = \cos 90^\circ + \cos 60^\circ = 0 + \frac{1}{2} = \frac{1}{2}$

When $C < D$ the factor formulas will produce negative angles. Sometimes the ratios could simply be reversed as in Example 5;
$\sin x + \sin 5x = \sin 5x + \sin x$. In the cases where this is not easily achieved, however, use the following useful results, which can be seen from the graphs for sine and cosine.



Note the rotational symmetry



Note the reflection in the y-axis

$\boldsymbol{\sin(-x) = -\sin x}$ ◀ Learn these results.

$\boldsymbol{\cos(-x) = \cos x}$

These results for negative angles can 'simplify' the factor formulas including negative angles.

Example 5

Solve:

a $\sin x + \sin 5x = \sin 3x$ for $0 \le x \le \pi$

b $\sin 3x - \sin 5x = \sin x$ for $0 \le x \le 180°$.

Solution

a Notice the three distinct angles, x, $3x$ and $5x$. The factor formula will help reduce this to a problem with fewer angles. The equation is easier to manipulate if rewritten as

$$\sin 5x + \sin x = \sin 3x$$

Then

$$2\sin\tfrac{1}{2}(5x+x)\cos\tfrac{1}{2}(5x-x) = \sin 3x$$

$$\Rightarrow \quad 2\sin 3x\cos 2x = \sin 3x$$

$$\Rightarrow \quad 2\sin 3x\cos 2x - \sin 3x = 0$$

$$\Rightarrow \quad \sin 3x(2\cos 2x - 1) = 0$$

Use the factor formu
for the sum of two si

Do not divide
throughout by sin 3x
you will lose solutio

Factorising.

So $\sin 3x = 0$ or $2\cos 2x - 1 = 0$

$\sin 3x = 0$ or $\cos 2x = \frac{1}{2}$

$$\sin 3x = 0 \Rightarrow 3x = 0, \pi, 2\pi, 3\pi, \ldots$$

$$\Rightarrow x = 0, \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \pi, \ldots$$

$$\cos 2x = \tfrac{1}{2} \Rightarrow 2x = \tfrac{\pi}{3}, \tfrac{5\pi}{3}, \tfrac{7\pi}{3}, \tfrac{11\pi}{3}, \ldots$$

$$\Rightarrow x = \tfrac{\pi}{6}, \tfrac{5\pi}{6}, \tfrac{7\pi}{6}, \tfrac{11\pi}{6}, \ldots$$

The values of x within the given range are: $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$.

Remember to use th
range in the questio

b Instead of rearranging $\sin 3x - \sin 5x = \sin x$ to make all terms positive (giving the equation $\sin 3x = \sin 5x + \sin x$, which we solved in **a**), we can apply the factor formulas to the original equation, and create a negative angle.

$$\sin 3x - \sin 5x = \sin x$$

$$\Rightarrow 2\cos 4x\sin(-x) = \sin x$$

Using the result $\sin(-x) = -\sin x$, gives

$$-2\cos 4x\sin x = \sin x$$

$$\Rightarrow 2\cos 4x\sin x + \sin x = 0$$

$$\Rightarrow \quad \sin x(2\cos 4x + 1) = 0$$

10.1 The Factor Formulas
Exercise
Technique

1 Solve the following equations for $0^\circ \le x \le 360^\circ$:

a $\sin x + \sin 2x + \sin 3x = 0$
b $\sin 5x - \sin x = 0$
c $\cos x + \cos 3x + \cos 5x = 0$
d $\sin 3x - \sin x = \cos 2x$

2 Solve the following equations for $0 \le x \le \pi$:

a $\sin x + \sin 2x - \sin 3x = 0$
b $\sin x + \sin 3x = 2 \sin 2x$
c $\cos 4x - \cos 2x + \sin 3x = 0$
d $\cos 5x - \cos x = 0$

3 Prove the following identities:

a $\sin\theta + \sin 2\theta + \sin 3\theta \equiv \sin 2\theta(1 + 2\cos\theta)$
b $\dfrac{\sin 3\theta + \sin 5\theta}{\sin 4\theta + \sin 6\theta} \equiv \dfrac{\sin 4\theta}{\sin 5\theta}$
c $\sin 3\theta + \sin\theta \equiv 4\sin\theta - 4\sin^3\theta$
d $\dfrac{\sin\theta + \sin 2\theta + \sin 3\theta}{\cos\theta + \cos 2\theta + \cos 3\theta} \equiv \tan 2\theta$
e $\dfrac{\cos 3\theta - \cos 5\theta}{\sin 4\theta} \equiv 2\sin\theta$

4 Write each of the following as the sum or difference of two trigonometrical functions:

a $2\sin 4\theta\cos 2\theta$
b $2\cos 5\theta\sin\theta$
c $2\cos 4\theta\cos 3\theta$
d $-2\sin 4\theta\sin 3\theta$
e $2\sin 4\theta\sin 2\theta$
f $2\sin 3\theta\sin 6\theta$

5 Without using a calculator find the exact value of the following:

a $\cos 105^\circ - \cos 15^\circ$
b $\sin 75^\circ + \sin 15^\circ$
c $2\sin 37\frac{1}{2}^\circ \sin 7\frac{1}{2}^\circ$
d $\sin 37\frac{1}{2}^\circ \cos 7\frac{1}{2}^\circ$

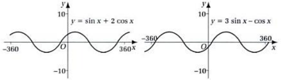
10.2 Functions of the form $f(x) = a \sin x + b \cos x$

Using a graphical calculator and making sure you are working in degree mode, set the range as follows.

$x_{\text{MIN}} = -360 \qquad x_{\text{MAX}} = 360 \qquad x_{\text{SCL}} = 60$

$y_{\text{MIN}} = -10 \qquad y_{\text{MAX}} = 10 \qquad y_{\text{SCL}} = 1$

Draw graphs of $y = a \sin x + b \cos x$ for various values of a and b. Try $a = 1, b = 2$ ($y = \sin x + 2 \cos x$) or $a = 3, b = -1$ ($y = 3 \sin x - \cos x$). What happens? The resulting graph in each case is a 'sine wave' or 'cosine wave'.



Each can be obtained from the graphs of $y = \sin x$ or $y = \cos x$ by performing two transformations:

- a translation parallel to the x-axis by some value (called the **phase angle**)
- a stretch parallel to the y-axis by some scale factor (called the **amplitude**).

These results suggest that functions of the form $f(x) = a \sin x + b \sin x$ can be written in the following forms:

$R \sin(x \pm \alpha)$ or $R \cos(x \pm \alpha)$,

By convention we always assume $R > 0$.

where α is the phase angle and R is the amplitude. The values of α and R depend on the values of a and b. This can be interpreted algebraically using the necessary compound angle formulas.

Suppose $\quad f(x) \equiv R \sin(x + \alpha)$
Then $\quad a \sin x + b \cos x \equiv R \sin(x + \alpha)$
So $\quad a \sin x + b \cos x \equiv R \sin x \cos \alpha + R \cos x \sin \alpha$
Now compare the coefficients of $\sin x$ and $\cos x$ on each side of this identity.

Recall the formula for $\sin(x + \alpha)$.

$\sin x: \quad a = R \cos \alpha \qquad [1]$

$\cos x: \quad b = R \sin \alpha \qquad [2]$

Since the values of a and b are known, the simultaneous equations [1] and [2] can be solved. Dividing equation [2] by equation [1] gives

$$\frac{b}{a} = \frac{R\sin\alpha}{R\cos\alpha}$$

$$\Rightarrow \frac{b}{a} = \tan\alpha$$

Notice that R will cancel.

So the phase angle $\boxed{\alpha = \tan^{-1}\left(\frac{b}{a}\right)}$ ◀ **Recall the alternative notation $\arctan\left(\frac{b}{a}\right)$.**

The principal value can be found using a calculator.

We can also find R. Squaring equations [1] and [2] and then adding them together gives,

$$\begin{aligned} a^2 + b^2 &= R^2\cos^2\alpha + R^2\sin^2\alpha \\ &= R^2(\sin^2\alpha + \cos^2\alpha) \\ &= R^2 \end{aligned}$$

By the Pythagorean identity, $\sin^2\theta + \cos^2\theta \equiv 1$.

So the amplitude $\boxed{R = \sqrt{a^2 + b^2}}$

Note the positive roo[t] $R > 0$.

Example 1

Write $f(x) = 3\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$ with $0^\circ \le \alpha \le 90^\circ$. Hence sketch the graph of $y = f(x)$ for $0^\circ \le x \le 360^\circ$.

Solution

Let $3\sin x + 4\cos x = R\sin(x+\alpha)$.

Then $3\sin x + 4\cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$

Equating coefficients of $\sin x$ and $\cos x$,

$$\sin x: \quad 3 = R\cos\alpha \qquad [1]$$

$$\cos x: \quad 4 = R\sin\alpha \qquad [2]$$

Dividing equation [2] by equation [1],

$$\tfrac{4}{3} = \tan\alpha$$

So $\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ$ (1 d.p.)

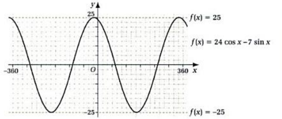
Find R by squaring and adding equations [1] and [2]:

$$\begin{aligned} 3^2 + 4^2 &= R^2\cos^2\alpha + R^2\sin^2\alpha \\ \Rightarrow 9 + 16 &= R^2(\sin^2\alpha + \cos^2\alpha) \\ \Rightarrow 25 &= R^2 \\ \Rightarrow R &= \sqrt{25} = 5 \end{aligned}$$

Recall that $R > 0$, so **ignore -5 as a soluti[on]**

We would usually differentiate $f(x)$ to find its maximum and minimum values, but in this case they can be found without differentiating. Consider the behaviour of the cosine function. It oscillates between a maximum value of $+1$ and a minimum value of -1. So $\cos(x + 16.3°)$ has a maximum value of $+1$ and a minimum value of -1, and since we have a stretch factor of 25, $f(x)$ has a maximum value of 25 and a minimum value of -25.

Remember to use radians when differentiating.



The equation $a \sin x + b \cos x = c$

Equations of the form $a \sin x + b \cos x = c$, where a, b and c are real numbers can be solved by first expressing the trigonometric terms as one function (either sine or cosine). The techniques discussed in Chapter 3 can then be applied and appropriate solutions to the equation identified within a given range of values for x.

Example 3

Express $12 \cos x + 5 \sin x$ in the form $R\cos(x - \alpha)$. Hence solve the equation $12 \cos x + 5 \sin x = 10$ for $0 \le x \le \pi$.

Solution

Let $\quad 12 \cos x + 5 \sin x = R\cos(x - \alpha)$.

Then $12 \cos x + 5 \sin x = R \cos x \cos \alpha + R \sin x \sin \alpha$

Equating coefficients of $\sin x$ and $\cos x$,

$$\sin x: \quad 5 = R \sin \alpha \qquad [1]$$

$$\cos x: \quad 12 = R \cos \alpha \qquad [2]$$

Dividing equation [1] by equation [2],

In this example we a working in radians, since they are used t specify the required range in the question

$$\tfrac{5}{12} = \tan \alpha$$

$$\Rightarrow \alpha = \tan^{-1}\left(\tfrac{5}{12}\right) = 0.395 \text{ (3 s.f.)}$$

Find R by squaring and adding equations [1] and [2]:

$$5^2 + 12^2 = R^2 \sin^2 \alpha + R^2 \cos^2 \alpha$$

$$\Rightarrow \quad 25 + 144 = R^2(\sin^2 \alpha + \cos^2 \alpha)$$

$$\Rightarrow \quad 169 = R^2$$

So $\quad R = \sqrt{169} = 13$

So $12\cos x + 5\sin x = 13\cos(x - 0.395)$.

We know that $12\cos x + 5\sin x = 10$. Substituting the new expression we have just found,

$$13\cos(x - 0.395) = 10$$

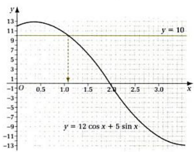
$$\cos(x - 0.395) = \tfrac{10}{13}$$

One possibility for x is given by

$$x - 0.395 = \cos^{-1}(\tfrac{10}{13}) = 0.693$$

So $\quad x = 0.693 + 0.395 = 1.088$

$$x = 1.09 \text{ (3 s.f.)}$$



Check that there is only one solution: $\cos(x - 0.395) = (\tfrac{10}{13})$ has other solutions, including -0.693 and 6.976, but they are outside the given range.

Maximising and minimising rational functions

Sometimes the denominator of a rational function contains terms involving $\sin x$ and $\cos x$. The technique of expressing these two terms as one trigonometric ratio can be useful in identifying the maximum and minimum values of the original function.

Example 4

Find the maximum and minimum values of $\dfrac{1}{15\cos x - 8\sin x + 23}$.

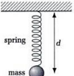
Periodic motion

Periodic motion is any motion that repeats itself in equal intervals of time. Equations used to model this type of motion contain trigonometric terms, because the sine and cosine functions are periodic. Examples of this type of motion include:

- a mass oscillating on the end of a spring;
- a buoy moving up and down on the waves on the surface of the water in a harbour;
- the tip of a sewing machine needle moving up and down.

A special case of this type of motion is known as '**simple harmonic motion**'.

Example 5



A mass is suspended from the end of a spring as shown in the diagram. The mass is oscillating. The distance d cm between the fixture point and the mass is given by

$$d = 17 + 12\sin 2t - 5\cos 2t,$$

where t seconds is the time after release. Find:

Notice that the angle is measured in radians; there is no degree symbol.

a the maximum and minimum distances from the fixture point reached by the mass

b the time at which the mass is first at its lowest point.

Solution

a Notice that the expression for d contains two trigonometric terms. $12\sin 2t - 5\cos 2t$ can be rewritten in the form $R\sin(2t - \alpha)$, where $0 \le \alpha \le \frac{\pi}{2}$.

$$\begin{aligned} 12\sin 2t - 5\cos 2t &\equiv R\sin(2t - \alpha) \\ &\equiv R\sin 2t\cos\alpha - R\cos 2t\sin\alpha \end{aligned}$$

Equating coefficients of $\sin 2t$ and $\cos 2t$,

$\sin 2t$: $\quad R\cos\alpha = 12 \qquad [1]$

$\cos 2t$: $\quad R\sin\alpha = 5 \qquad [2]$

Contextual

1 The depth of water in a leaking storage tank is d cm at time t hours after midnight on Sunday. The value of d is given by $d = 10 - 2\sqrt{3}\cos 4t^\circ - 2\sin 4t^\circ$. Find:

a the least and greatest depth of water possible with this model

b the time at which the depth first reaches these values.

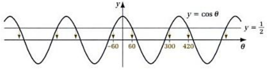
2 A mass is suspended vertically from the end of a spring. It is allowed to oscillate so that the distance of the mass from the fixture point, d cm, is given by $d = 23 - 5\sqrt{3}\sin 3t - 5\cos 3t$, where t seconds is the time after release. Find:

a the minimum and maximum distances from the fixture point reached by the mass

b the times at which these points are first reached by the mass.

10.3 General Solutions of Trigonometric Equations

Solve the equation $\cos\theta = \frac{1}{2}$. Now check what you have written. A calculator in degree mode gives an answer of $60°$, but recall that this is only the principal solution. Check that $\theta = 300°$, $\theta = 420°$ and $\theta = -60°$ also work. The equation $\cos\theta = \frac{1}{2}$ had no restriction on the range of values of θ that are acceptable as solutions. A check of this equation graphically demonstrates that there is an infinite number of solutions.



These solutions can, however, be written concisely in one algebraic statement. Notice that all of the solutions can be found by adding multiples of $360°$ to either $60°$ or $-60°$. This means that

$\theta = 360°n \pm 60°$, where $n \in \mathbb{Z}$

$\mathbb{Z}$ is the set of all integers (both positive and negative).

describes all of the possible solutions. The statement $\theta = 360°n \pm 60°$ is the **general solution** to the equation $\cos\theta = \frac{1}{2}$. Substitute appropriate values of n into the general equation to verify that $\theta = 300°$, $\theta = 420°$ and $\theta = -60°$ can be derived from it.

Look again at the structure of the general solution. Notice that it has two distinct parts.

- $360°n$ multiples of $360°$ ($n \in \mathbb{Z}$)
- $\pm 60°$ the principal solution.

This means that the general solution to the equation $\cos\theta = k$ where $-1 \leq k \leq 1$ can be found by

- finding the principal solution by evaluating $\cos^{-1} k$; and
- adding a term to include multiples of $360°$ or 2π radians.

Example 1

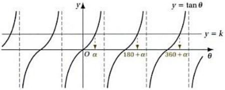
Find the general solution, in degrees, of the equation $\cos\theta = 0.3$.

Solution

The principal solution is $\theta = \cos^{-1}(0.3) = 72.5°$

The general solution is $\theta = 360°n \pm 72.5°$, where $n \in \mathbb{Z}$

Now consider the equation $\tan\theta = k$ where $k \in \mathbb{R}$. Notice that all of the solutions can be found by adding on multiples of 180° (or π) to the principal solution α. This means that the general solution to $\tan\theta = k$, where $k \in \mathbb{R}$, has the form $\theta = 180°n + \alpha$.

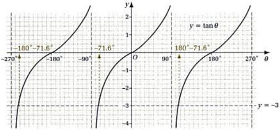


Example 4

Find the general solution, in degrees, of the equation $\tan\theta = -3$.

Solution

The principal solution is $\theta = \tan^{-1}(-3) = -71.6°$.
The general solution is $\theta = 180°n - 71.6°$, where $n \in \mathbb{Z}$.
Check this solution graphically.



General solutions can also be found in equations involving double angles, multiple angles and both the sine and cosine functions.

Example 5

Find the general solution to the equation $\cos\theta = \cos 2\theta$.

Solution

One way of solving this would be to expand the double angle using $\cos 2\theta = 2\cos^2\theta - 1$. This creates a quadratic in $\cos\theta$ that can be solved.

Check this by expanding $\cos 2\theta$ and solving the resulting quadratic.

An alternative method is to allow the expression 2θ to be the principal solution.

If $\quad \cos\theta = \cos 2\theta$

then $\quad \theta = 2\pi n \pm 2\theta; n \in \mathbb{Z}$

Notice that the secor set of solutions, $\theta = -2\pi n$, is include the first set because multiples of $\frac{2}{3}\pi$ inclu the multiples of 2π.

Adding and subtracting 2θ on both sides,

$$3\theta = 2\pi n \quad \text{or} \quad -\theta = 2\pi n$$

$$\theta = \frac{2\pi n}{3} \quad \text{or} \quad \theta = -2\pi n$$

So the general solution is $\theta = \frac{2}{3}\pi n$, where $n \in \mathbb{Z}$.

Example 6

Find the general solution of the equation $3\cos\theta + 4\sin\theta = 1$.

Solution

Notice that this equation is of the form $a\cos\theta + b\sin\theta = c$. The LHS of the equation can be rewritten using an expression of the form $R\cos(\theta - \alpha)$. Check that

$$3\cos\theta + 4\sin\theta = 5\cos(\theta - 53.1^\circ)$$

Now $3\cos\theta + 4\sin\theta = 1 \Rightarrow 5\cos(\theta - 53.1^\circ) = 1$

$$\text{So } \cos(\theta - 53.1^\circ) = \tfrac{1}{5}$$

The principal solution is given by $\theta - 53.1^\circ = \cos^{-1}(\frac{1}{5}) = 78.5^\circ$.
So $\theta - 53.1^\circ = 360^\circ n \pm 78.5^\circ, n \in \mathbb{Z}$.
The general solution for θ is in two parts.

$$\theta = 360^\circ n + 78.5^\circ + 53.1^\circ$$

$$\Rightarrow \quad \theta = 360^\circ n + 131.6^\circ$$

$$\text{and} \quad \theta = 360^\circ n - 78.5^\circ + 53.1^\circ$$

$$\Rightarrow \quad \theta = 360^\circ n - 25.4^\circ, \quad \text{where } n \in \mathbb{Z}$$

10.3 General Solutions of Trigonometric Equations

Exercise

Technique

1 Find the general solutions, in degrees, of the following equations:

a $\cos\theta = 0.9$
b $\cos\theta = -0.25$
c $\sin\theta = 0.2$
d $\sin\theta = -0.6$
e $\tan\theta = 0.5$
f $\tan\theta = -1.5$

2 Find the general solutions of the following equations:

a $\cos\theta = \frac{1}{2}$
b $\cos\theta = -\frac{\sqrt{3}}{2}$
c $\sin\theta = \frac{\sqrt{3}}{2}$
d $\sin\theta = -1$
e $\tan\theta = 1$
f $\tan\theta = 0$

3 Find the general solutions of the following equations:

a $\sin 2\theta = \frac{1}{2}$
b $\cos\theta = \cos 3\theta$
c $\sin\theta = \sin 3\theta$
d $\cos 4\theta = -\frac{1}{2}$

4 Find the general solution, in degrees, of the equation $\cos\theta + 3\sin\theta = 2$.

5 Find the general solution of the equation $\cos\theta + \cos 3\theta + \cos 5\theta = 0$.

6 Find the general solution of the equation $\sin(x + 30^\circ) = \cos(x + 45^\circ)$.

Consolidation

Exercise A

1 Solve the equation $\cos\theta + \cos 5\theta = 2\cos 2\theta$ for $0 \leq \theta \leq 2\pi$.

2 Express $63\sin x + 16\cos x$ in the form $R\sin(x + \alpha)$, where R is positive and α is acute. Find:

a the acute angle x for which $63\sin x + 16\cos x = 50$

b the obtuse angle x for which $63\sin x + 16\cos x = 0$.

(UCLES)

3 Express $4\sin\theta - 3\cos\theta$ in the form $R\sin(\theta - \alpha)$, where R is positive and α is an acute angle. Hence, or otherwise, find the greatest and least values of the expression

$$\frac{1}{(10 - 3\cos\theta + 4\sin\theta)}.$$

(NICCEA)

4 Prove the identity $\cos 2\theta - \cos 4\theta \equiv 2\cos^2\theta - 2\cos^2 2\theta$. By substituting $\theta = 36°$, show, without using a calculator, that $\cos 36° - \cos 72° = \frac{1}{2}$. Hence find the value of $\cos 36°$ in the form $a + b\sqrt{5}$, where a and b are to be found.

(WJEC)

5 Express $3\sin\theta - 4\cos\theta$ in the form $R\sin(\theta - \alpha)$ where $R > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$. Give the value of α in radians to three decimal places. Determine the greatest and least values of the following:

a $(3\sin\theta - 4\cos\theta)^2$ **b** $\dfrac{1}{(3\sin\theta - 4\cos\theta)^2 + 1}$

State a value of θ in radians for which the least value of the expression in **b** occurs.

(AEB)

6 **a** Express the function $3\cos x° + 4\sin x°$ in the form $R\sin(x + \alpha)°$, stating the values of R and α.

b Write down the maximum value of $3\cos x° + 4\sin x°$.

c Solve the equation $3\cos x° + 4\sin x° = 0$ for $0° \leq x \leq 360°$.

(NEAB)

7 Express $\sqrt{3}\sin\theta - \cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Hence, or otherwise, find all values of θ, for $0° \leq \theta \leq 360°$, which satisfy the equation $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$.

(UCLES)

8 Prove the identity $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 \equiv 2 + 2\cos(A - B)$.
Solve the equation $(\cos 4\theta + \cos\theta)^2 + (\sin 4\theta + \sin\theta)^2 = 2\sqrt{3}\sin 3\theta$, giving the general solution in degrees.

(AEB)

9 **a** Rewrite $2\cos x - \sin x$ in the form $R\cos(x + \alpha)$, where R is real and α is acute.

b Hence find the general solution of the equation $2\cos x - \sin x = 1$.

(NICCEA)

Exercise B

1 Solve the equation $\sin 8\theta - \sin 2\theta = 0$ for $0 \le \theta \le \frac{\pi}{2}$.

2 Find all values of θ lying between $0°$ and $360°$ satisfying the equation $4\sin\theta + 3\cos\theta = 1$. Give your answers correct to the nearest degree.

(WJEC)

3 Given that $4\cos\theta + 3\sin\theta \equiv R\cos(\theta - \alpha)$, find the value of R and the value of α where $R > 0$ and $0° < \alpha < 90°$.

a Hence find all values of θ between $0°$ and $360°$ satisfying the equations:

i $4\cos\theta + 3\sin\theta = 2$

ii $4\cos 2\theta + 3\sin 2\theta = 5\cos\theta$.

b Find the greatest and least values of the expression $\dfrac{1}{(4\cos\theta + 3\sin\theta + 6)}$, and give the corresponding values of θ between $0°$ and $360°$.

(WJEC)

4 The function f is defined for all real values of x by
$f(x) = (\cos x - \sin x)(17\cos x - 7\sin x)$.

a By first multiplying out the brackets, show that $f(x)$ may be expressed in the form $5\cos 2x - 12\sin 2x + k$, where k is a constant. State the value of k.

b Given that $5\cos 2x - 12\sin 2x \equiv R\cos(2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, state the value of R and find the value of α in radians to three decimal places.

c Determine the greatest and least values of $\dfrac{39}{f(x) + 14}$, and state a value of x at which the greatest value occurs.

(AEB)

5 Express $2\cos\theta + 2\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the values of R and α in exact form. Hence, or otherwise, show that one of the acute angles θ satisfying the equation $2\cos\theta + 2\sin\theta = \sqrt{6}$ is $\frac{5\pi}{12}$, and find the other acute angle.

(UCLES)

6 [In this question, give your answers correct to the nearest 0.1°, where appropriate.]

a Find the general solution, in degrees, of the equation $5\cos 2\theta = 3$.

b Solve each of the following equations, for $0 < \theta < 180°$:

i $5\cos\theta + 2\sin^2\theta = 4$

ii $5\sin\theta + 3\cos\theta = 5$

(UCLES)

7 By squaring both sides of the identity $\sin^2\theta + \cos^2\theta \equiv 1$, prove that $4(\sin^4\theta + \cos^4\theta) \equiv 3 + \cos 4\theta$. Find the general solution in radians of the equation $4(\sin^4\theta + \cos^4\theta) = 2 - \cos 2\theta$.

(AEB)

Applications and Activities

1 *The mathematical representation of a wave*

The general equation for a wave represents every point x on the wave at every time t,

$$y = A\sin(kx - \omega t),$$

where A is the amplitude of the wave, k is the wave number and ω is the angular frequency (velocity). Then k is the number of waves contained in the interval of 2π.

Now consider sound as an example of a wave. If two waves y_1 and y_2 given by

$$y_1 = A_1 \sin(k_1 x - \omega_1 t) \quad \text{and} \quad y_2 = A_2 \sin(k_2 x - \omega_2 t)$$

act at the same time, the resultant sound (from the **principle of superposition**) is given by $y_1 + y_2$. Investigate the behaviour of $y_1 + y_2$, and particularly the analysis of the resultant wave in the real world. How is this interpreted?

2 *Simple harmonic motion (SHM)*

SHM is motion in which the acceleration of a body is directly proportional to its displacement from the equilibrium position but in the opposite direction. Investigate SHM when the displacement of the object performing the motion is written in the form

$$x = A\cos\omega t + B\sin\omega t,$$

where ω is the angular velocity, t is time and A and B are constants to be determined. What is an advantage of writing displacement in this form?

Summary

- The factor formulas, used to convert sums and differences of sines and cosines into products, are:

$$\sin C + \sin D \equiv 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\sin C - \sin D \equiv 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$\cos C + \cos D \equiv 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\cos C - \cos D \equiv -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

- The function $a\sin x + b\cos x$ can be expressed as:

  $R\sin(x+\alpha)$ $\quad$ $R\sin(x-\alpha)$ $\quad$ $R\cos(x+\alpha)$ $\quad$ $R\cos(x-\alpha)$

  depending on the sign values of a and b. R and α can be found by first expanding using the appropriate compound angle formula, and then equating coefficients to produce two simultaneous equations. Squaring and adding them gives $R = \sqrt{a^2+b^2}$ and dividing them gives an equation for $\tan\alpha$.
  This technique is useful for

  1. solving equations of the form $a\sin x + b\cos x = c$;
  2. finding maximum and minimum values of rational functions with trigonometric denominators;
  3. analysing periodic motion.

- Trigonometric equations of the form:

  $\sin\theta = k$ and $\cos\theta = k$, where $-1 \le k \le 1$ and $\tan\theta = k$, where $k \in \mathbb{R}$

  can be solved to give both a principal solution and a general solution. The principal solution can be found using a calculator. The general solution formula can then be used to generate other valid solutions as required. Remember that the general solution can be written in both degree and radian form.

  1. If $\cos\theta = k$, then $\theta = 360^\circ n \pm \alpha$ or $\theta = 2\pi n \pm \alpha$.
  2. If $\sin\theta = k$, then $\theta = 180^\circ n + (-1)^n\alpha$ or $\theta = \pi n + (-1)^n\alpha$.
  3. If $\tan\theta = k$ then $\theta = 180^\circ n + \alpha$ or $\theta = \pi n + \alpha$.

11 Differentiation II

What you need to know

- How to differentiate rational powers of x, and sine and cosine functions.
- How to use the chain rule to differentiate composite functions.
- How to locate stationary points on a curve and determine their nature by looking at the sign of $\frac{dy}{dx}$ on either side of the stationary point.
- How to find the equation of the tangent and the normal to a curve at a given point.
- How to solve equations involving e^x and $\ln x$.
- How to write down the Cartesian equation of a circle.
- How to use the binomial theorem to express $(1 + ax)^n$ as a series of ascending powers of x, where n is rational.

Review

1 Differentiate each of the following with respect to x:

a $y = x^2 - 9x + 11$
b $y = 10\sqrt{x}$
c $y = x^{\frac{7}{2}} - x^{\frac{5}{2}}$
d $y = 3x + \frac{1}{2x}$
e $y = \sin 4x$
f $y = 3\sin^2 x$
g $y = \cos(\frac{1}{2}x + \frac{1}{4}\pi)$
h $y = 2x - \cos 2x$

2 Use the chain rule to find $\frac{dy}{dx}$ for each of the following:

a $y = (x + 11)^4$
b $y = (4x - 1)^8$
c $y = \sqrt{x + 5}$
d $y = \sqrt{2 - x^2}$
e $y = \frac{6}{3x+1}$
f $y = \frac{4}{\sqrt{x+2}}$

3 Find the coordinates of the stationary points on the graph of $y = \frac{1}{4}x^4 - x^3 - 3x^2 + 8x + 3$ and determine their nature.

Hint: Use the factor theorem to factorise $\frac{dy}{dx}$.

4 Find the equations of the tangent and the normal to the curve $y = \frac{4}{x}$ at the point where $x = 2$.

5 Find the exact solutions of the following exponential and logarithmic equations:

a $e^{x-3} = 4$
b $\ln(5x) = -3$
c $e^{4x} = 10$
d $\ln(x - 1) = 6$

6 Write down the Cartesian equation for each of the following circles:

a radius 4, centre (0, 0) **b** radius 6, centre (0, 3)
c radius 7, centre (4, −2) **d** radius $\sqrt{3}$, centre (−1, 6)

7 Use the binomial theorem to express each of the following as a series of ascending powers of x, up to and including the x^3 term:

a $(1+x)^{-2}$ **b** $(1-x)^{\frac{1}{3}}$
c $\dfrac{1}{1-3x}$ **d** $\sqrt{1+4x}$

11.1 Differentiating Products and Quotients

In Chapter 5, the techniques for differentiating polynomials, functions involving rational powers of x, and sine and cosine functions were developed. Many other mathematical functions are formed by multiplying or dividing two or more of these types of functions. Some of these products and quotients can be differentiated by first expressing the function as a polynomial, and then differentiating term by term.

However, there are many products and quotients, such as $y = (2x+3)(x-1)^4$ and $y = \frac{(3x+1)}{(x-2)}$, where it is either difficult or impossible to express the overall function as a polynomial. Functions of this type can be differentiated using two standard results, or algorithms, which are known as the **product** and **quotient rules**.

The product rule

Suppose $y = u(x)v(x)$ is the product of two separate functions of x; u and v. Any small change, δx, in the value of x will give rise to corresponding small changes, δu and δv, in the values of functions u and v respectively. These in turn result in a small change, δy, in the value of y, such that

$$\begin{aligned}\delta y &= (y+\delta y) - y\\ &= u(x+\delta x)v(x+\delta x) - u(x)v(x)\end{aligned}$$

Since $y + \delta y = u(x+\delta x)v(x+\delta x)$ is the value of the function corresponding to $x + \delta x$.

It is possible to rewrite this expression for δy in the form

$$\delta y = u(x+\delta x)v(x+\delta x) - u(x+\delta x)v(x) + v(x)u(x+\delta x) - u(x)v(x)$$

Adding the terms $-u(x+\delta x)v(x)$ and $v(x)u(x+\delta x)$. Notice that these terms cancel.

It follows then that

$$\delta y = u(x+\delta x)[v(x+\delta x) - v(x)] + v(x)[u(x+\delta x) - u(x)]$$

Differentiating from first principles,

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)\\ &= \lim_{\delta x\to 0}\left[u(x+\delta x)\left(\frac{v(x+\delta x)-v(x)}{\delta x}\right)\right]\\ &\quad + \lim_{\delta x\to 0}\left[v(x)\left(\frac{u(x+\delta x)-u(x)}{\delta x}\right)\right]\\ &= \lim_{\delta x\to 0}[u(x+\delta x)] \times \lim_{\delta x\to 0}\left[\frac{v(x+\delta x)-v(x)}{\delta x}\right]\\ &\quad + v(x) \times \lim_{\delta x\to 0}\left[\frac{u(x+\delta x)-u(x)}{\delta x}\right]\\ &= u(x) \times \frac{dv}{dx} + v(x) \times \frac{du}{dx}\end{aligned}$$

As $\delta x \to 0$,

$$u(x+\delta x) \to u(x)$$

$$\frac{v(x+\delta x)-v(x)}{\delta x} \to \frac{dv}{dx}$$

$$\frac{u(x+\delta x)-u(x)}{\delta x} \to \frac{du}{dx}$$

The **product rule** for differentiating functions of the form $y = u(x)v(x)$ can be stated as

If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

Using the abbreviations u' and v' for $\frac{du}{dx}$ and $\frac{dv}{dx}$ respectively, the product rule is more commonly stated as

If $y = uv$, then $\dfrac{dy}{dx} = uv' + vu'$ ◀ **Learn this important result.**

Example 1

Using the product rule, differentiate $y = (2x + 3)(x - 1)^4$ with respect to x.

Solution

Let $y = uv$ where $u = (2x + 3)$ and $v = (x - 1)^4$. Then $u' = 2$ and $v' = 4(x - 1)^3$.

Remember to use the chain rule to find v'.

Using the product rule,

$$\begin{aligned}\frac{dy}{dx} &= uv' + vu' \\ &= (2x + 3) \times 4(x - 1)^3 + (x - 1)^4 \times 2 \\ &= 4(2x + 3)(x - 1)^3 + 2(x - 1)^4 \\ &= 2(x - 1)^3[2(2x + 3) + (x - 1)] \\ &= 2(x - 1)^3(5x + 5) \\ &= 10(x + 1)(x - 1)^3\end{aligned}$$

Spot the common fac $2(x - 1)^3$.

Factorise the express as much as possible.

An advantage of the product rule is that it is usually possible to get a factorised expression for $\frac{dy}{dx}$. This is particularly useful when trying to locate and determine the nature of any stationary points.

Example 2

Find the gradient of the curve $y = x^2 \cos x$ when $x = \pi$.

Solution

Let $y = uv$ where $u = x^2$ and $v = \cos x$. Then $u' = 2x$ and $v' = -\sin x$.

Using the product rule,

$$\begin{aligned}\frac{dy}{dx} &= uv' + vu' \\ &= x^2 \times (-\sin x) + \cos x \times 2x \\ &= -x^2 \sin x + 2x \cos x \\ &= x(2\cos x - x \sin x)\end{aligned}$$

When $x = \pi$, the gradient of the curve $y = x^2 \cos x$ is

$$\frac{dy}{dx} = \pi(2\cos\pi - \pi\sin\pi)$$
$$= -2\pi$$

Recall the special angles: $\cos\pi = -1$ and $\sin\pi = 0$.

The quotient rule

Suppose $y = \frac{u(x)}{v(x)}$ is the quotient of two separate functions of x; u and v. This can be rewritten as the product of u and $\frac{1}{v}$, such that

$$y = \frac{u}{v} = u \times \frac{1}{v}$$

Using the product rule to differentiate $y = u \times \frac{1}{v}$ gives

$$\frac{dy}{dx} = u\frac{d}{dx}\left(\frac{1}{v}\right) + \frac{1}{v}\frac{du}{dx}$$
$$= u \times -\frac{1}{v^2}\frac{dv}{dx} + \frac{1}{v}\frac{du}{dx}$$
$$= -\frac{u}{v^2}\frac{dv}{dx} + \frac{v}{v^2}\frac{du}{dx}$$
$$= \frac{1}{v^2}\left[-u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

Using the chain rule,

$$\frac{d}{dx}\left(\frac{1}{v}\right) = \frac{d}{dv}\left(\frac{1}{v}\right) \times \frac{dv}{dx}$$
$$= -\frac{1}{v^2}\frac{dv}{dx}$$

The common denominator is v^2.

So we have the **quotient rule** for differentiating functions of the form $y = \frac{u(x)}{v(x)}$.

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{1}{v^2}\left[v\frac{du}{dx} - u\frac{dv}{dx}\right]$

This is more commonly written as

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$ ◀ **Learn this important result.**

Example 3

Use the quotient rule to differentiate the following functions with respect to x:

a $y = \frac{3x+1}{x-2}$ **b** $y = \frac{\sqrt{x}}{2x+5}$ **c** $y = \frac{\sin 2x}{x^2}$

Solution

a Let $y = \frac{u}{v}$, where $u = 3x + 1$ and $v = x - 2$. Then $u' = 3$ and $v' = 1$.

11.1 Differentiating Products and Quotients

Exercise

Technique

1 Use the product and quotient rules to differentiate each of the following with respect to x:

a $y = x(x+4)^6$

b $y = (4x+3)(x+1)^4$

c $y = (x-6)\sqrt{x-1}$

d $y = x^3 \sin 2x$

e $y = \frac{2x+1}{x-3}$

f $y = \frac{3x}{(x-2)^2}$

g $y = \frac{\sin x}{2x}$

h $y = \frac{\sin 3x}{\cos 2x}$

1 c, d

2 Find the gradient of each of the following curves at the points indicated:

a $y = 3x(x-4)^3$ at $(5, 15)$

b $y = (x-3)(x+2)^4$ at $(-3, -6)$

c $y = \frac{x+5}{2x+7}$ at $(1, \frac{2}{3})$

d $y = \frac{x^3}{x-2}$ at $(-2, 2)$

e $y = (2x-3)\sin x$ at $(\frac{\pi}{2}, \pi - 3)$

f $y = (1-2x)\sqrt{x}$ at $(4, -14)$

Contextual

1 Find the equations of the tangent and the normal to the following curves at the points indicated:

a $y = \frac{x+2}{2x-5}$ at $(2, -4)$

b $y = (x-2)(x+4)^3$ at $(-3, -5)$

c $y = x^2 \cos x$ at $(\pi, -\pi^2)$

d $y = \frac{x^2}{x^2+3}$ at $(-1, \frac{1}{4})$

	$a=2$	$a=2.5$	$a=2.7$	$a=2.8$	$a=3$	$a=4$
$\delta x=0.1$	0.717 73				1.161 23	1.486 98
$\delta x=0.01$	0.695 56				1.104 67	1.395 95
$\delta x=0.001$	0.693 39				1.099 22	1.387 26
$\delta x=0.0001$	0.693 17				1.098 67	1.386 39
$\delta x=0.000\,01$	0.693 15				1.098 62	1.386 30
limit as $\delta x \to 0$	0.693 15				1.098 61	1.386 29

Notice that as $\delta x \to 0$, the factor $\frac{a^{\delta x}-1}{\delta x}$ converges to a limiting value, which is different for each value of a. For example,

$$\frac{\mathrm{d}}{\mathrm{d}x}(2^x) \approx 0.69315 \times 2^x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(3^x) \approx 1.09861 \times 3^x$$

This suggests that there is a value of a between 2 and 3 for which $\lim\limits_{\delta x \to 0}\left(\frac{a^{\delta x}-1}{\delta x}\right) = 1$. For this particular value of a, it follows that $\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x$.

Use a calculator to complete the above table of values of $\frac{a^{\delta x}-1}{\delta x}$ for $a = 2.5$, 2.7 and 2.8. By further trial and improvement, find the value of a, correct to three decimal places, for which $\lim\limits_{\delta x \to 0}\left(\frac{a^{\delta x}-1}{\delta x}\right) = 1$.

This value of a is an irrational number. Its value is $\mathrm{e} = 2.718\,281\,8$, correct to seven decimal places. It follows that

$$\lim_{\delta x \to 0}\left(\frac{\mathrm{e}^{\delta x}-1}{\delta x}\right) = 1.$$

This means that the derivative of *the* exponential function $y = \mathrm{e}^x$ is

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x$$

◄ **Learn this important result.**

The exponential function is the only function that remains unaltered when differentiated.

An alternative way of establishing this result is to use the series expansion of e^x introduced in Chapter 7:

$$\mathrm{e}^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Differentiating this infinite power series term by term,

$$\begin{aligned}\frac{d}{dx}(e^x) &= 0 + \frac{1}{1!} + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots \\ &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ &= e^x\end{aligned}$$

It can also be shown that the derivative of any exponential function of the form $y = a^x$ is

$$\frac{d}{dx}(a^x) = a^x \ln a$$ ◀ **Learn this important result.**

This confirms that $\frac{d}{dx}(e^x) = e^x$, because $\ln e = 1$.

This means that $\lim_{\delta x \to 0}\left(\frac{a^{\delta x}-1}{\delta x}\right) = \ln a$.

The chain rule can be used to differentiate exponential functions of the form $y = e^{f(x)}$, where $f(x)$ is some function of x.
Let $y = e^u$, where $u = f(x)$. Then $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = f'(x)$

Using the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u \times f'(x) \\ &= f'(x)e^{f(x)}\end{aligned}$$ ◀ **Derivative of the power × original exponential function**

Example 1

Differentiate each of the following with respect to x:

a $y = e^{3x+1}$
b $y = e^{\sin x}$
c $y = 3^{2x}$

Solution

a Let $y = e^u$ where $u = 3x + 1$. Then $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 3$.

Using the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u \times 3 \\ &= 3e^{3x+1}\end{aligned}$$

Using the quotient rule,

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$
$$= \frac{3e^x - (3x+2)e^x}{(e^x)^2}$$
$$= \frac{(1-3x)e^x}{(e^x)^2}$$
$$= \frac{1-3x}{e^x}$$

c Let $x = uv$, where $u = e^{2t}$ and $v = \cos 3t$. Then $u' = \frac{du}{dt} = 2e^{2t}$ and $v' = \frac{dv}{dt} = -3\sin 3t$.

Using the product rule,

$$\frac{dx}{dt} = uv' + vu'$$
$$= -3e^{2t}\sin 3t + 2e^{2t}\cos 3t$$
$$= e^{2t}(2\cos 3t - 3\sin 3t)$$

Common factor of e^{2t}

Example 3

As a result of a slump in the housing market, a property initially valued at £60 000 on 1 January depreciates. Its value t weeks later can be modelled by $V = 60\,000e^{kt}$ where its value is £V and k is a constant to be determined. Exactly one year later, the property remains unsold and is valued at only £50 000.

a Find an expression for k.

b Find the market value of the property after 26 weeks to the nearest pound.

c Find an expression for $\frac{dV}{dt}$. Then find the rate at which the property's value is depreciating at the start and end of the year to the nearest pound.

Solution

a After one year, when $t = 52$, $V = 50\,000$.
Substituting into the expression for V gives

$$50\,000 = 60\,000e^{52k}$$
$$\Rightarrow \quad e^{52k} = \tfrac{5}{6}$$
$$\Rightarrow \quad 52k = \ln\left(\tfrac{5}{6}\right)$$
$$\Rightarrow \quad k = \tfrac{1}{52}\ln\left(\tfrac{5}{6}\right)$$

Take logarithms of both sides.

$k \approx -0.0035$; a negative value is expected for depreciation.

b When $t = 26$

$$V = 60\,000e^{26k}$$
$$= 60\,000e^{\frac{1}{2}\ln\left(\frac{5}{6}\right)}$$
$$= £54\,772$$

c Given that $V = 60\,000e^{kt}$, where $k = \frac{1}{52}\ln(\frac{5}{6})$, differentiate V with respect to t.

$$\frac{dV}{dt} = 60\,000ke^{kt}$$

$$= \frac{60\,000}{52}\ln(\tfrac{5}{6})e^{\frac{t}{52}\ln(\frac{5}{6})}$$

At the start of the year, $t = 0$. The rate at which the property's value is depreciating is found by putting $t = 0$ into this expression.
Check that

$$\frac{dV}{dt} = \frac{60\,000}{52}\ln(\tfrac{5}{6})e^{0}$$

$$= -210.37$$

To the nearest pound, the rate of depreciation is £210 per week.
Carry out a similar calculation by substituting $t = 52$. Verify that after one year, the rate of depreciation is £175 per week

Logarithmic functions

To differentiate the natural logarithmic function $y = \ln x$ from first principles, consider the small change, δy, in the value of this function that results from a small change, δx, in the value of x.

$$\text{Since}\quad \delta y = (y + \delta y) - y$$

$$= \ln(x + \delta x) - \ln x,$$

where $y + \delta y = \ln(x + \delta x)$ is the value of the function corresponding to $x + \delta x$, it follows that

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$$

$$= \lim_{\delta x \to 0}\left[\frac{\ln(x + \delta x) - \ln x}{\delta x}\right]$$

Unfortunately there is no easy way of expanding the $\ln(x + \delta x)$ term in this expression. An alternative approach must be taken in order to find $\frac{dy}{dx}$.
Recall that the natural logarithmic function $y = \ln x$ is the inverse of the exponential function $y = e^x$.
If $y = \ln x$, then $x = e^y$.
Differentiating both sides with respect to y,

$$\frac{dx}{dy} = e^y$$

Example 4

Differentiate the following with respect to x:

a $y = \ln(x^2 + 3x - 2)$

b $y = \ln(\cos x)$

c $y = x \ln x$

d $y = \frac{(\ln x)}{e^x}$

e $y = \log_{10}(x^2 + 1)$

Solution

a $y = \ln(x^2 + 3x - 2) \Rightarrow \frac{dy}{dx} = \frac{2x+3}{x^2+3x-2}$

Use $\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$.

b $y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$

c Let $y = uv$, where $u = x$ and $v = \ln x$. Then $u' = 1$ and $v' = \frac{1}{x}$.

Using the product rule, $\frac{dy}{dx} = uv' + vu'$

$$= x\left(\frac{1}{x}\right) + \ln x$$

$$= 1 + \ln x$$

d Let $y = \frac{u}{v}$ where $u = \ln x$ and $v = e^x$. Then $u' = \frac{1}{x}$ and $v' = e^x$.

Using the quotient rule, $\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$$= \frac{1}{(e^x)^2}\left[e^x\left(\frac{1}{x}\right) - e^x \ln x\right]$$

Common factor e^x.

$$= \frac{1}{e^x}\left(\frac{1}{x} - \ln x\right)$$

Take out factor $\frac{1}{x}$.

$$= \frac{1}{e^x} \times \frac{1}{x}(1 - x\ln x)$$

$$= \frac{(1 - x\ln x)}{xe^x}$$

e Let $y = \log_{10} u$, where $u = (x^2 + 1)$. Then $\frac{dy}{du} = \frac{1}{u \ln 10}$ and $\frac{du}{dx} = 2x$.

Using the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \frac{1}{u \ln 10} \times 2x$$

$$= \frac{2x}{(x^2 + 1)\ln 10}$$

11.2 Differentiation of Exponentials and Logarithms

Exercise

Technique

1 Differentiate each of the following with respect to x:

1 f

a	$y = \mathrm{e}^{2x-3}$	**f**	$y = x^2\mathrm{e}^{-x}$
b	$y = \mathrm{e}^{x^3}$	**g**	$y = \mathrm{e}^{-x/2}\sin x$
c	$y = \mathrm{e}^{1-5x}$	**h**	$y = \frac{\mathrm{e}^x+1}{x^2}$
d	$y = \mathrm{e}^{\cos x}$	**i**	$y = \mathrm{e}^x \sin 2x$
e	$y = 4^{-x}$	**j**	$y = \frac{\cos^2 x}{\mathrm{e}^x}$

2 Differentiate each of the following with respect to x:

2 f, h

a	$y = \ln(4x)$	**f**	$y = \ln(\sin x)$
b	$y = \ln(x^4)$	**g**	$y = x^2 \ln x$
c	$y = \ln(x^5)$	**h**	$y = \frac{\ln x}{x^2}$
d	$y = \ln(4x+1)$	**i**	$y = \ln(1+x) - \ln(1-x)$
e	$y = \ln(x^4+1)$	**j**	$y = \log_{10}(3x-4)$

3 For each of the following curves:

i find expressions for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

ii locate and determine the nature of any stationary points.

a	$y = (2x+1)\mathrm{e}^x$	**b**	$y = \ln(x+2) + \frac{1}{(x+2)}$
c	$y = \mathrm{e}^x \sin x \ (-\pi \le x \le \pi)$	**d**	$y = (x+1)^3\mathrm{e}^{-x}$

Contextual

1 Find the equations of the tangent and the normal to the following curves at the points indicated:

a	$y = \ln(3x+1)$ at $x = 1$	**b**	$y = 3\mathrm{e}^{2x+3}$ at $x = -1$
c	$y = x^2 \ln x$ at $x = \mathrm{e}$	**d**	$y = (x+1)\mathrm{e}^{-x}$ at $x = 2$

2 The population, P, of a new town development grows exponentially for the first 25 years such that $P = 1000 + 200\mathrm{e}^{0.13t}$, where t is the number of years since its establishment.

a What is the initial population of the town?

b What is its population after 10 and 20 years?

c Find an expression for the rate at which the population increases at any time t. Use this to calculate the rate of increase after 10 and 20 years.

d Calculate, to the nearest month, how long after its establishment it is before the rate of increase in the population reaches 200 people per year.

3 A ball-bearing is released from rest from the surface of a large tank of oil into which it drops. After t seconds it has reached a depth d centimetres, where $d = 6t - 4e^{-1.5t} + 4$.

a Calculate the depth of the ball-bearing to the nearest millimetre, after 1, 2 and 3 seconds.

b Find expressions for the velocity, v, and acceleration, a, of the ball-bearing after t seconds

c Calculate its velocity and acceleration after 0.5 seconds

d Explain what happens to the acceleration and velocity as t becomes large.

4 The price £P of a particular laptop computer t weeks after its release is given by $P = 1100 + 3t - 30\ln(t + 2)$.

a After how many weeks does the price reach its lowest value? What is the minimum price?

b How much is the laptop after six months and what is the rate at which P is changing at this time?

Solution

a Let $y = \tan u$, where $u = 2x$. Then $\frac{dy}{du} = \sec^2 u$ and $\frac{du}{dx} = 2$.

Using the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 2\sec^2 u$

$= 2\sec^2 2x$

b Let $y = \cot u$, where $u = (x^3 + 2)$. Then $\frac{dy}{du} = -\text{cosec}^2 u$ and $\frac{du}{dx} = 3x^2$.

Using the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= -\text{cosec}^2 u \times 3x^2$

$= -3x^2 \text{cosec}^2 (x^3 + 2)$

c Let $y = 2u^3$, where $u = \tan x$. Then $\frac{dy}{du} = 6u^2$ and $\frac{du}{dx} = \sec^2 x$.

Using the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 6u^2 \times \sec^2 x$

$= 6\tan^2 x \sec^2 x$

Differentiating cosec and sec

The derivatives of the two other reciprocal trigonometric functions, $\text{cosec}\, x = \frac{1}{\sin x}$ and $\sec x = \frac{1}{\cos x}$, can also be found using the chain rule.

Let $y = \frac{1}{u}$, where $u = \sin x$. Then $\frac{dy}{du} = -\frac{1}{u^2}$ and $\frac{du}{dx} = \cos x$.

Using the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= -\frac{1}{u^2} \times \cos x$

$= -\frac{\cos x}{\sin^2 x}$

$= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x}$

$= -\text{cosec}\, x \cot x$

So $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$ ◀ **Learn this result.**

Now find the derivative of $\sec x$.

Let $y = \frac{1}{u}$, where $u = \cos x$. Then $\frac{dy}{du} = -\frac{1}{u^2}$ and $\frac{du}{dx} = -\sin x$

Example 3

Find the derivatives of each of the following functions:

a $y = 2x\tan x$ **b** $y = \frac{\operatorname{cosec} x}{x}$ **c** $y = \tan 3x \sec x$

Solution

a Let $y = uv$, where $u = 2x$ and $v = \tan x$. Then $u' = 2$ and $v' = \sec^2 x$.

Using the product rule,
$$\begin{aligned}\frac{dy}{dx} &= uv' + vu' \\ &= 2x\sec^2 x + 2\tan x \\ &= 2(x\sec^2 x + \tan x)\end{aligned}$$

b Let $y = \frac{u}{v}$ where $u = \operatorname{cosec} x$ and $v = x$. Then $u' = -\operatorname{cosec} x \cot x$ and $v' = 1$.

Using the quotient rule,
$$\begin{aligned}\frac{dy}{dx} &= \frac{vu' - uv'}{v^2} \\ &= \frac{-x\operatorname{cosec} x\cot x - \operatorname{cosec} x}{x^2} \\ &= \frac{-\operatorname{cosec} x(x\cot x + 1)}{x^2}\end{aligned}$$

c Let $y = uv$, where $u = \tan 3x$ and $v = \sec x$. Then $u' = 3\sec^2 3x$ and $v' = \sec x \tan x$.

Using the chain rule to differentiate tan 3x.

Using the product rule,
$$\begin{aligned}\frac{dy}{dx} &= uv' + vu' \\ &= \tan 3x \sec x \tan x + 3\sec^2 3x \sec x \\ &= \sec x(\tan 3x \tan x + 3\sec^2 3x)\end{aligned}$$

Differentiating inverse trigonometric functions

If $y = \sin^{-1} x$, then $x = \sin y$. Differentiating both sides of this equation *with respect to y* gives

$$\frac{dx}{dy} = \cos y \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$ ◀ **Using** $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$.

Now,
$$\begin{aligned}&\cos^2 y + \sin^2 y \equiv 1 \\ \Rightarrow\ &\cos^2 y = 1 - \sin^2 y \\ &\qquad\;\; = 1 - x^2\end{aligned}$$

That is, $\cos y = \sqrt{1 - x^2}$

Notice that only the positive root is taken. This is because $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$, so $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, giving $\cos y \ge 0$.

So $$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$ ◀ **Learn this result.**

d Let $y = \tan^{-1} u$, where $u = \frac{x}{a}$. Then $\frac{dy}{du} = \frac{1}{1+u^2}$ and $\frac{du}{dx} = \frac{1}{a}$.

Using the chain rule,
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{1+u^2} \times \frac{1}{a}$$
$$= \frac{1}{a\left(1 + \frac{x^2}{a^2}\right)}$$
$$= \frac{a}{a^2 + x^2}$$

Multiply top and bottom by a.

Example 5

Use the product and quotient rules to differentiate each of the following:

a $y = x \sin^{-1} x$ **b** $y = \frac{\tan^{-1} x}{1+x^2}$

Solution

a Let $y = uv$, where $u = x$ and $v = \sin^{-1} x$. Then $u' = 1$ and $v' = \frac{1}{\sqrt{1-x^2}}$.

Using the product rule
$$\frac{dy}{dx} = uv' + vu'$$
$$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x$$

b Let $y = \frac{u}{v}$, where $u = \tan^{-1} x$ and $v = 1 + x^2$. Then $u' = \frac{1}{1+x^2}$ and $v' = 2x$.

Using the quotient rule,
$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$
$$= \frac{\left(\frac{1+x^2}{1+x^2}\right) - 2x\tan^{-1} x}{(1+x^2)^2}$$
$$= \frac{1 - 2x\tan^{-1} x}{(1+x^2)^2}$$

11.3 Further Trigonometric Differentiation

Exercise

Technique

1 Differentiate each of the following with respect to x:

1 a

a $y = \tan 7x$
b $y = \sec 3x$
c $y = \cot(x^2)$
d $y = \operatorname{cosec}^2 5x$
e $y = \cos^{-1} 9x$
f $y = \sin^{-1}(1 - x)$
g $y = \tan^{-1} x^2$
h $y = \tan^{-1}(2x + 3)$

2 Use the product and quotient rules to differentiate each of the following with respect to x:

2 a

a $y = x^2 \tan x$
b $y = 5x \sec x$
c $y = 2x \sin^{-1} x$
d $y = x^3 \tan^{-1} x$

Contextual

1 Find the gradient of each of the following curves at the point indicated and the equations of the tangent and the normal at that point.

a $y = \tan(\frac{x}{3})$ at $(\pi, \sqrt{3})$
b $y = \sin^{-1}(1 - x)$ at $(\frac{1}{2}, \frac{\pi}{6})$
c $y = x \sec x$ at $(\pi, -\pi)$

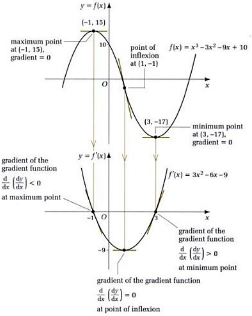
11.4 Using the Second Derivative

Stationary points

Recall from Chapter 5 that the stationary points on a graph can be located by solving the equation $\frac{dy}{dx} = 0$. Their nature can be determined by looking at the gradient of the graph on either side of each stationary point. It is also possible to use the second derivative to help decide whether a particular stationary point is a maximum or a minimum point.

The gradient of a curved graph is itself a function of x. By drawing the graphs of $y = f(x)$ and its gradient function $y = f'(x)$, it is possible to relate features that appear on the two graphs.

For example, sketch the graphs of the function $f(x) = x^3 - 3x^2 - 9x + 10$ and its derivative $f'(x) = 3x^2 - 6x - 9$. Check the nature of the stationary points as you draw the graphs. Check your results using a graphical calculator.



Notice that at the maximum point on the curve $y = x^3 - 3x^2 - 9x + 10$, the function changes from being an increasing function of x to a decreasing function of x. This 'negative' change in the gradient, from positive through zero to negative, as the curve passes through $(-1, 15)$, means that the 'gradient of the gradient function', or **second derivative**, is negative at the maximum point.

So at the maximum point on the curve, $\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$.

At a maximum point, $\dfrac{d^2y}{dx^2} < 0$ ◄ **Remember this important result.**

At the minimum point on the curve $y = x^3 - 3x^2 - 9x + 10$, the function changes from being a decreasing function of x to an increasing function of x. The corresponding change in the gradient is from negative through zero to positive as the curve passes through $(3, -17)$. This indicates that the 'gradient of the gradient function', or second derivative, is positive at the minimum point.

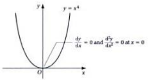
So at the minimum point on the curve, $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$.

At a minimum point, $\dfrac{d^2y}{dx^2} > 0$ ◄ **Remember this important result.**

There is also a point of inflexion at $(1, -1)$ on the curve $y = x^3 - 3x^2 - 9x + 10$. This is where the tangent to the curve crosses from one side to another. This is not a stationary point. Instead it is the point at which the gradient of the curve is at its most negative. The 'gradient of the gradient function', or second derivative, is zero at this point of inflexion.

In fact $\frac{d^2y}{dx^2} = 0$ at all points of inflexion, whether they are stationary or not. It is possible, however, for the second derivative to equal zero at points that *are not* points of inflexion. Consider the graph of $y = x^4$. This has a stationary point at $x = 0$. Its second derivative $\frac{d^2y}{dx^2} = 12x^2$ equals zero when $x = 0$, but this stationary point is clearly a minimum.

Check this with a graphical calculato
Check also the grap
$y = -x^4$. This is a maximum at $x = 0$ where $\frac{d^2y}{dx^2} = 0$.



The only reliable way of determining the nature of a stationary point at which $\frac{d^2y}{dx^2} = 0$ is to look at the gradient, $\frac{dy}{dx}$, on either side of the point.

Summary

- $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ at maximum points
- $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ at minimum points
- if $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, the sign of the gradient, $\frac{dy}{dx}$, on either side of the stationary point must then be found to determine its nature – **do not assume that it is a stationary point of inflexion.**

Example 1

Find the coordinates of the stationary points on the curve $y = (x + 3)(x - 2)^4$. Use the second derivative to determine their nature. Sketch the curve.

Solution

Let $y = uv$ where $u = (x + 3)$ and $v = (x - 2)^4$. Then $u' = 1$ and $v' = 4(x - 2)^3$.

Use the chain rule to differentiate $(x - 2)^4$.

Using the product rule, $\frac{dy}{dx} = uv' + vu'$

$$= (x + 3) \times 4(x - 2)^3 + (x - 2)^4 \times 1$$
$$= 4(x + 3)(x - 2)^3 + (x - 2)^4$$
$$= (x - 2)^3[4(x + 3) + (x - 2)]$$
$$= (x - 2)^3(5x + 10)$$
$$= 5(x + 2)(x - 2)^3$$

At stationary points on the curve, $\frac{dy}{dx} = 0$.

$$5(x + 2)(x - 2)^3 = 0$$
$$\Rightarrow \quad x = -2 \text{ or } x = 2$$

Verify that when $x = -2$, $y = 256$ and that when $x = 2$, $y = 0$.
The stationary points are located at $(-2, 256)$ and $(2, 0)$. Their nature can be determined by evaluating the second derivative at $x = -2$ and $x = 2$ respectively.

Use the product rule again to find $\frac{d^2y}{dx^2}$.
Let $\frac{dy}{dx} = fg$, where $f = 5(x + 2)$ and $g = (x - 2)^3$. Then $f' = 5$ and $g' = 3(x - 2)^2$.
It follows that

Use the chain rule to differentiate $(x - 2)^3$.

f and g have been used here instead of u and v to avoid confusion with the earlier working.

$$\frac{d^2y}{dx^2} = fg' + gf'$$
$$= 15(x + 2)(x - 2)^2 + 5(x - 2)^3$$
$$= 5(x - 2)^2[3(x + 2) + (x - 2)]$$
$$= 20(x - 2)^2(x + 1)$$

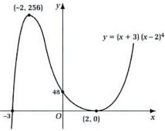
When $x = -2$, $\frac{d^2y}{dx^2} = 20(-4)^2(-1) = -320 < 0$.
So $(-2, 256)$ is a maximum point.

When $x = 2$, $\frac{d^2y}{dx^2} = 20(0)^2(3) = 0$.
Looking at the gradient of the curve on either side of $x = 2$:

- when $x = 1$, $\frac{dy}{dx} = 5(3)(-1)^3 = -15$
- when $x = 3$, $\frac{dy}{dx} = 5(5)(1)^3 = 25$

The gradient changes from negative to positive as the curve passes through $x = 2$, so $(2, 0)$ is a minimum point.

This information can now be used to sketch the curve.

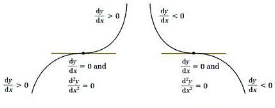


◀ Check this result using a graphical calculator.

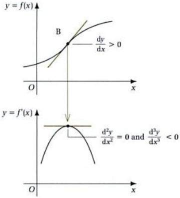
Check that the curve crosses or touches the axes at $(-3, 0)$, $(2, 0)$ and $(0, 48)$.

General points of inflexion

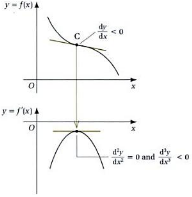
Stationary points of inflexion have two forms. The gradient of the curve can be positive on both sides of a stationary point of inflexion or negative on both sides, as shown in the diagram. Check that a tangent drawn to the curve at a stationary point of inflexion crosses from one side to the other.



However stationary points of inflexion are special cases. The gradient of a curve does not have to be zero at a point of inflexion. In fact, there must be at least one non-stationary point of inflexion on any smooth continuous curve between two stationary points. This situation can best be described graphically.



The gradient reaches a maximum value at point B. This means $\frac{d^2y}{dx^2} = 0$. Since the gradient of the gradient function is changing from positive to negative as it passes through B, $\frac{d^3y}{dx^3} < 0$ at B.



The gradient of the curve is negative on either side of the point C. Its value is least negative (that is at its maximum) at point C. This means $\frac{d^2y}{dx^2} = 0$ at this point. The gradient of the gradient function is changing from positive to negative, so the third derivative $\frac{d^3y}{dx^3} < 0$.

11.4 Using the Second Derivative

Exercise

Technique

1 For each of the following cubic functions:

i find expressions for $f'(x)$ and $f''(x)$

ii find the coordinates of any stationary points on the graph of $y = f(x)$ and use the second derivative to determine their nature

iii sketch the graph of $y = f(x)$.

a $f(x) = x^3 - 12x^2 + 45x - 40$

b $f(x) = 7 - 12x + 6x^2 - x^3$

c $f(x) = 2x^3 - 9x^2 - 108x + 40$

2 **a** State the coordinates of the points where the curve $y = (x-5)(x+3)^3$ crosses the axes.

b Locate and determine the nature of the stationary points on the curve $y = (x-5)(x+3)^3$.

Hint: Use the product rule to differentiate.

c Find the coordinates of the non-stationary point of inflexion on this curve. Use the first and third derivatives to determine its nature.

d Sketch $y = (x-5)(x+3)^3$.

3 Locate and determine the nature of the maximum and minimum points on the graph $y = x^4 - 24x^2 + 32$. Show that there are two non-stationary points of inflexion on this curve, at $x = -a$ and $x = a$, where a is a positive integer to be determined.

4 **a** Find the coordinates of the stationary points on the curve $y = x + 2\cos x$ in the interval $-2\pi \leq x \leq 2\pi$. Use the second derivative to determine their nature.

b Find the coordinates of the points of inflexion on the curve in this interval.

c Sketch $y = x + 2\cos x$ for $-2\pi \leq x \leq 2\pi$.

5 For each of the following curves:

i write down the coordinates of the points where the curve crosses the axes

ii write down the equation of any vertical asymptotes

iii find an expression for $\frac{dy}{dx}$

iv locate and determine the nature of any stationary points

v sketch the curve.

a $y = x(x-1)^4$

b $y = (x+4)^2(x-2)^2$

c $y = \frac{x^2}{(2x+6)}$

6 For each of the following curves:

i find expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$;

ii locate and determine the nature of any stationary points.

a $y = (2x + 1)e^x$

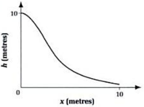
b $y = \ln(x + 2) + \dfrac{1}{(x + 2)}$

c $y = e^x \sin x$ (for $-\pi \le x \le \pi$)

d $y = (x + 1)^3 e^{-x}$

Contextual

1



The height h metres above the ground of part of a funfair ride, shown above, can be modelled using $h = 5(x + 2)e^{-x/2}$.

a Find expressions for $\frac{dh}{dx}$ and $\frac{d^2h}{dx^2}$.

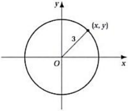
b Calculate the value of x at which this section of the ride is steepest, and the height above the ground at this point.

c Calculate this maximum gradient. At what angle, to the nearest degree, is the track to the ground at this point?

11.5 Implicit Differentiation

All of the functions that have been differentiated so far have been expressed in the form $y = f(x)$. However there are curves in the x–y plane, with equations linking x and y, that cannot be written in this way. Some examples are circles, ellipses and hyperbolae. The equations for these curves are called **implicit functions**. The method used to find the gradient at any point on their curves is called **implicit differentiation**.

For example, the Cartesian equation of a circle of radius 3, centred at the origin, is $x^2 + y^2 = 9$.

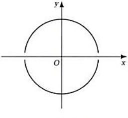


Rearranging the equation,

$$y^2 = 9 - x^2$$

$$y = \pm\sqrt{9 - x^2}$$

Notice that two explicit functions $y = +\sqrt{9 - x^2}$ and $y = -\sqrt{9 - x^2}$ are needed to completely define this circle. Using a graphical calculator, plot $y = +\sqrt{9 - x^2}$ and $y = -\sqrt{9 - x^2}$. Check that this draws a circle, centred at the origin of radius 3.



The limited resolution of your graphical calculator may result in the circle appearing to have been incompletely drawn.

Both of these explicit functions can be differentiated using the chain rule. Implicit differentiation allows us to find an expression for $\frac{dy}{dx}$ from the original equation.
Differentiating each term in the equation $x^2 + y^2 = 9$ with respect to x,

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(9) \qquad [1]$$

We know that $\frac{d}{dx}(x^2) = 2x$, and $\frac{d}{dx}(9) = 0$.
Use the chain rule to change the variable with which the y^2 term is differentiated:

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx}$$

$$\Rightarrow \frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$$ ◄ **Learn this important result.**

From equation [1], these results give

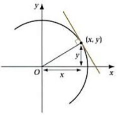
$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

This result is true for all circles centred at the origin, because the radius squared term in the equation of the circle is zero when differentiated. This can be confirmed by considering the radial line from the centre O to some point (x, y) on the circumference of the circle.

Notice that the gradient of this line is $\frac{y}{x}$. Because it also intercepts the circle at right angles, it is the normal to the circle at this point. This means that the gradient of the tangent to the circle at this point is $-\frac{x}{y}$.



Remember that the gradient of the tanger is also the gradient of the curve (circle) itsel at this point.

Example 1

A circle C is centred at $(6, -2)$ and has a radius of 5 units.

a Write down the Cartesian equation of this circle

b Find an expression for $\frac{dy}{dx}$.

c Find the gradient of the circle at the two points where $x = 9$.

Solution

a Recall that, for a circle of radius r, centred at (a, b),

$$(x-a)^2 + (y-b)^2 = r^2$$

The Cartesian equation of this circle is therefore

$$(x-6)^2 + (y+2)^2 = 25$$

Verify that the equivalent form is $x^2 + y^2 - 12x + 4y + 15$

b Differentiating both sides of the equation with respect to x,

$$\frac{d}{dx}(x-6)^2 + \frac{d}{dx}(y+2)^2 = \frac{d}{dx}(25)$$

Alternatively, differenti $x^2 + y^2 - 12x + 4y + 15$ term by term, using tl **chain rule for the terr in y.**

Using the chain rule, $\frac{d}{dx}(x-6)^2 = 2(x-6)$

and $$\frac{d}{dx}(y+2)^2 = \frac{d}{dy}(y+2)^2 \times \frac{dy}{dx}$$

$$= 2(y+2)\frac{dy}{dx}$$

So $$2(x-6) + 2(y+2)\frac{dy}{dx} = 0$$

Then $$2(y+2)\frac{dy}{dx} = -2(x-6)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(6-x)}{(y+2)}$$

c Substitute $x = 9$ into $(x-6)^2 + (y+2)^2 = 25$.

This gives $$3^2 + (y+2)^2 = 25$$

$$9 + y^2 + 4y + 4 = 25$$

$$y^2 + 4y - 12 = 0$$

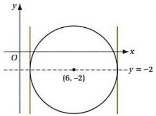
$$(y+6)(y-2) = 0$$

$$\Rightarrow y = -6 \text{ or } y = 2.$$

At $(9, -6)$, $\frac{dy}{dx} = \frac{6-9}{-6+2} = \frac{-3}{-4} = \frac{3}{4}$

At $(9, 2)$, $\frac{dy}{dx} = \frac{6-9}{2+2} = -\frac{3}{4}$

Notice that in this example, $\frac{dy}{dx} = \frac{6-x}{y+2}$ is undefined when $y = -2$. This corresponds to the two points on the circle where the tangents are parallel to the y-axis.



The denominator in $\frac{6-x}{y+2}$ is zero when $y = -2$.

Unlike the equation of a circle, many implicit functions have equations that include product terms. These are expressions such as xy, xy^2 and x^2y. These terms can still be differentiated implicitly but require the use of the product rule.

Example 2

For each of the following, find an expression for $\frac{dy}{dx}$ in terms of x and y.

a $x^2 + 3xy - 4x = 9$ **b** $x^2y^2 + 4x^2 - y^2 = 0$

Solution

a Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(3xy) - \frac{d}{dx}(4x) = \frac{d}{dx}(9)$$

Use the product rule to differentiate the second term. Let $u = 3x$ and $v = y$. Then $u' = 3$ and $v' = \frac{dy}{dx}$.

Now $\frac{d}{dx}(3xy) = uv' + vu' = 3x\frac{dy}{dx} + 3y$

It follows that differentiation of both sides of the equation gives

$$2x + 3x\frac{dy}{dx} + 3y - 4 = 0$$

Rearranging to find $\frac{dy}{dx}$,

$$3x\frac{dy}{dx} = 4 - 2x - 3y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4 - 2x - 3y}{3x}$$

b Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}(x^2y^2) + 8x - \frac{d}{dx}(y^2) = 0$$

Using the product rule, $\frac{d}{dx}(x^2y^2) = 2x^2y\frac{dy}{dx} + 2xy^2$

and using the chain rule $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$

So $2x^2y\frac{dy}{dx} + 2xy^2 + 8x - 2y\frac{dy}{dx} = 0$

$$\Rightarrow \quad (2x^2y - 2y)\frac{dy}{dx} = -(2xy^2 + 8x)$$

$$\Rightarrow \quad 2y(x^2 - 1)\frac{dy}{dx} = -2x(y^2 + 4)$$

Eliminate the common factor 2.

$$\Rightarrow \quad y(1 - x^2)\frac{dy}{dx} = x(y^2 + 4)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x(y^2 + 4)}{y(1 - x^2)}$$

Now factorise and simplify the equation.

$$2 - 8\frac{dy}{dx} + (4 - 4x)\frac{d^2y}{dx^2} = 0$$

$$2 - 8\left(\frac{2y - x}{2 - 2x}\right) + (4 - 4x)\frac{d^2y}{dx^2} = 0$$

Verify that this equation can be rearranged to give

$$4(1 - x)\frac{d^2y}{dx^2} = \frac{16y - 4x - 4}{2(1 - x)}$$

$$\frac{d^2y}{dx^2} = \frac{4y - x - 1}{2(1 - x)^2}$$

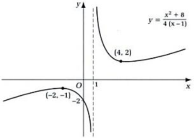
When $x = -2$ and $y = -1$, $\frac{d^2y}{dx^2} = -\frac{3}{18} = -\frac{1}{6} < 0$

So $(-2, -1)$ is a maximum point on the curve.

When $x = 4$ and $y = 2$, $\frac{d^2y}{dx^2} = \frac{3}{18} = \frac{1}{6} > 0$

So $(4, 2)$ is a minimum point on the curve.

Notice that the equation of the curve in this example can be rearranged into the form $y = f(x)$. Show that $y = \frac{x^2+8}{4(x-1)}$. Its first and second derivatives can then be found using the quotient rule.
Use a graphical calculator to draw the graph of $y = \frac{x^2+8}{4(x-1)}$. Confirm that the curve has a maximum point at $(-2, -1)$ and a minimum point at $(4, 2)$.



11.5 Implicit Differentiation Exercise

Technique

1 For each of the following circles:

i write down its Cartesian equation
ii find an expression for $\frac{dy}{dx}$ in terms of x and y
iii calculate the gradient of the circle at the point P, whose coordinates are indicated:

a radius 5, centre $(1, 2)$; P $(5, 5)$
b radius 13, centre $(3, -4)$; P$(-2, 8)$
c radius $\sqrt{39}$, centre $(5, 0)$; P $(-1, \sqrt{3})$

2 Find an expression for $\frac{dy}{dx}$ in terms of x and y for each of the following:

a $3x^2 - 2y^2 + xy = 0$
b $x^3 + 4y - 3xy = 0$
c $6x^2 + 2y^2 - y + 4 = 0$
d $4y^3 - 5x + 2xy + 6 = 0$
e $2x + 7y - 3x^2y - 4 = 0$
f $5x - 3y^2 + 2xy^2 + 1 = 0$

3 Find the gradient of each of the following curves at the point indicated:

a $y^3 - 2x^3 - 4xy + 24 = 0$ at $(2, 2)$
b $4y - 3x^2 - y^2 + 17 = 0$ at $(2, 5)$
c $3y^2 - 3x - x^2y - 47 = 0$ at $(-1, 4)$

4 For each of the following curves:

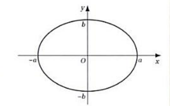
i find an expression for $\frac{dy}{dx}$ in terms of x and y
ii find the coordinates of the stationary points(s) on the curve
iii find an expression for $\frac{d^2y}{dx^2}$ in terms of x, y and $\frac{dy}{dx}$
iv determine the nature of the stationary point(s):

a $3x^2 + y^2 - 3xy - 9 = 0$
b $5x^2 + 2y^2 - 2y - 12 = 0$
c $x + 12y + x^2y + 2 = 0$

5 Find the equations of the tangent and the normal to each of the following curves at the points indicated: 5 c

a $2x + 3y - xy^2 + 4 = 0$ at $(-1, -2)$
b $x + 2y^2 + 4xy^2 - 12 = 0$ at $(2, 1)$
c $y^3 + 10x + xy - 15 = 0$ at $(6, -3)$

Example 1**b** is particularly important. The parametric equations of an ellipse centred at the origin are $x = a\cos\theta$, $y = b\sin\theta$. Eliminating the parameter θ gives the Cartesian equation for the ellipse as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



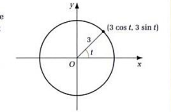
The Cartesian equation of a curve can usually be obtained from parametric equations. Eliminate the parameter, t, from the two parametric equations, $x = f(t)$ and $y = g(t)$. How is the gradient of such a parametrically defined curve found? First differentiate both x and y with respect to the parameter t, to find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ respectively. Then, using the chain rule

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$, which can be rewritten as

Recall that $\frac{dx}{dt} = \frac{1}{\left(\frac{dt}{dx}\right)}$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right)$$ ◀ **Learn this result.**

For example, check that the parametric equations that define the circle of radius 3, centred at the origin, are $x = 3\cos t$ and $y = 3\sin t$. The parameter, t, is the angle, in radians, measured anticlockwise about the origin from the x-axis.



This parameter can be eliminated by squaring and adding the equations for x and y. Check that

$$\begin{aligned} x^2 + y^2 &= (3\cos t)^2 + (3\sin t)^2 \\ &= 9\cos^2 t + 9\sin^2 t \\ &= 9(\cos^2 t + \sin^2 t) \\ &= 9 \end{aligned}$$

◀ $\cos^2\theta + \sin^2\theta \equiv 1$

Thus $x^2 + y^2 = 9$, which is the Cartesian equation of this circle.

Differentiating the two parametric equations $x = 3\cos t$ and $y = 3\sin t$ with respect to t gives

$$\frac{dx}{dt} = -3\sin t \quad \text{and} \quad \frac{dy}{dt} = 3\cos t.$$

Using the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

$$= \left(\frac{dy}{dt}\right) \bigg/ \left(\frac{dx}{dt}\right)$$

$$= \frac{3\cos t}{-3\sin t} = -\frac{1}{\tan t}$$

$\dfrac{dt}{dx} = \dfrac{1}{\left(\frac{dx}{dt}\right)}$

Notice that since $x = 3\cos t$ and $y = 3\sin t$,

$$\frac{dy}{dx} = \frac{3\cos t}{-3\sin t} = -\frac{x}{y}$$

This was the result obtained by implicitly differentiating the Cartesian equation of this circle in the previous section.

Example 2

Find an expression for $\frac{dy}{dx}$ for each of the following parametrically defined curves:

a $x = 4t^2 + 1$, $y = t^3 + t$ **b** $x = 3\sin\theta$, $y = \cos 2\theta$

Solution

a $\frac{dx}{dt} = 8t$ and $\frac{dy}{dt} = 3t^2 + 1$

So $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt}\right) \bigg/ \left(\dfrac{dx}{dt}\right) = \dfrac{3t^2+1}{8t}$

b $\frac{dx}{d\theta} = 3\cos\theta$ and $\frac{dy}{d\theta} = -2\sin 2\theta$

So $\dfrac{dy}{dx} = \left(\dfrac{dy}{d\theta}\right) \bigg/ \left(\dfrac{dx}{d\theta}\right)$

$$= -\frac{2\sin 2\theta}{3\cos\theta}$$ ◄ $\sin 2\theta = 2\sin\theta\cos\theta$

$$= -\frac{4\sin\theta\cos\theta}{3\cos\theta} = -\tfrac{4}{3}\sin\theta$$

Example 3

Find the coordinates of the point(s) on the following parametrically defined curves where the gradient has the value indicated.

a $x = \dfrac{4}{t}, y = \dfrac{8}{t^2}, \dfrac{dy}{dx} = \dfrac{2}{3}$ **b** $x = \sin\theta, y = 2\cos^3\theta, \dfrac{dy}{dx} = -3$

Solution

a $\dfrac{dx}{dt} = \dfrac{-4}{t^2}$ and $\dfrac{dy}{dt} = \dfrac{-16}{t^3}$

So $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt}\right) \bigg/ \left(\dfrac{dx}{dt}\right) = \left(\dfrac{-16}{t^3}\right) \bigg/ \left(\dfrac{-4}{t^2}\right) = \left(\dfrac{-16}{t^3}\right) \times \left(\dfrac{-t^2}{4}\right) = \dfrac{4}{t}$

If $\dfrac{dy}{dx} = \dfrac{2}{3}$, then $\dfrac{4}{t} = \dfrac{2}{3} \Rightarrow t = 6$

When $t = 6$, $x = \frac{4}{6} = \frac{2}{3}$ and $y = \frac{8}{36} = \frac{2}{9}$
So $\frac{dy}{dx} = \frac{2}{3}$ at $(\frac{2}{3}, \frac{2}{9})$

b $\frac{dx}{d\theta} = \cos\theta$ and $\frac{dy}{d\theta} = -6\cos^2\theta\sin\theta$

So $\dfrac{dy}{dx} = \left(\dfrac{dy}{d\theta}\right) \Big/ \left(\dfrac{dx}{d\theta}\right) = \dfrac{-6\cos^2\theta\sin\theta}{\cos\theta} = -6\cos\theta\sin\theta = -3\sin 2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

If $\frac{dy}{dx} = -3$, then $-3\sin 2\theta = -3$

So $\sin 2\theta = 1$

Then $2\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \ldots$

and $\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \ldots$

When $\theta = \frac{\pi}{4}$, $x = \frac{1}{\sqrt{2}}$ and $y = 2(\frac{1}{\sqrt{2}})^3 = \frac{1}{\sqrt{2}}$.

When $\theta = \frac{5\pi}{4}$, $x = -\frac{1}{\sqrt{2}}$ and $y = 2(-\frac{1}{\sqrt{2}})^3 = -\frac{1}{\sqrt{2}}$.

$\theta = \frac{9\pi}{4}$ will repeat coordinates

So $\frac{dy}{dx} = -3$ at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

It is possible to find a general equation for the tangent, or the normal, at some general point $(x = f(t), y = g(t))$ on a parametric curve. The equation of the tangent, or normal, at a specific point on the curve can then be found by substituting an appropriate value of the parameter t.

Example 4

The curve C has parametric equations $x = t^3$ and $y = 6t$. Find a general equation for both the tangent and the normal at some point $(t^3, 6t)$ on the curve. Write down the equation of the tangent to the curve at $(8, 12)$ and the equation of the normal to the curve at $(-1, -6)$.

Solution

$\dfrac{dx}{dt} = 3t^2$ and $\dfrac{dy}{dt} = 6$

So $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt}\right) \Big/ \left(\dfrac{dx}{dt}\right) = \dfrac{6}{3t^2} = \dfrac{2}{t^2}$

The gradient of the tangent, and the normal, to the curve at some general point $(t^3, 6t)$ are $\frac{2}{t^2}$ and $-\frac{1}{2}t^2$, respectively. Use $y - y_1 = m(x - x_1)$ to find the equations of these lines.

The equation of the tangent is

$$y - 6t = \frac{2}{t^2}(x - t^3)$$

Multiply both sides by t^2

$$\Rightarrow \quad t^2y - 6t^3 = 2x - 2t^3$$

$$\Rightarrow \quad t^2y - 2x = 4t^3 \quad \text{or} \quad t^2y - 2x - 4t^3 = 0$$

The equation of the normal is

$$y - 6t = -\tfrac{1}{2}t^2(x - t^3)$$
$$\Rightarrow \quad 2y - 12t = -t^2x + t^5$$
$$\Rightarrow \quad 2y + t^2x = t^5 + 12t \quad \text{or} \quad 2y + t^2x - t^5 - 12t = 0$$

Multiply both sides b

At the point (8, 12), $t = 2$. The equation of the tangent at this point, found by substituting $t = 2$ into $t^2y - 2x = 4t^3$, is

$$4y - 2x = 32 \quad \text{or} \quad 2y - x - 16 = 0$$

At the point $(-1, -6)$, $t = -1$. Substituting $t = -1$ into the general equation for a normal to the curve $2y + t^2x = t^5 + 12t$, gives

$$2y + x = -13 \quad \text{or} \quad 2y + x + 13 = 0$$

Stationary points and second derivatives

The second derivative $\frac{d^2y}{dx^2}$ of a parametrically defined curve is found by differentiating the first derivative $\frac{dy}{dx}$ with respect to x (that is, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$). But, for most parameter curves, $\frac{dy}{dx}$ will be a function of the parameter, t, and *not* x. This means the chain rule must be used to change the variable by which $\frac{dy}{dx}$ is being differentiated, so that

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

Having found an expression for $\frac{d^2y}{dx^2}$, it can be used to determine the nature of any stationary points on the parameter curve.

Example 5

Find an expression for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following parametrically defined curves:

a $x = 2t + 3$
$y = \frac{5}{t}$

b $x = 4\cos\theta$
$y = \sin^2\theta$

Solution

a $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = \frac{-5}{t^2}$

$$\text{So} \quad \frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right) = \frac{-5}{2t^2}$$

Now $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

$= \frac{d}{dt}\left(\frac{-5}{2t^2}\right) \times \frac{dt}{dx}$

$= \frac{5}{t^3} \times \frac{1}{2}$

$= \frac{5}{2t^3}$

Remember that $\frac{dt}{dx} = \frac{1}{\left(\frac{dx}{dt}\right)} = \frac{1}{2}$ in this case.

b $\frac{dx}{d\theta} = -4\sin\theta$ and $\frac{dy}{d\theta} = 2\sin\theta\cos\theta$

So $\frac{dy}{dx} = \left(\frac{dy}{d\theta}\right) \Big/ \left(\frac{dx}{d\theta}\right)$

$= \frac{2\sin\theta\cos\theta}{-4\sin\theta}$

$= -\frac{1}{2}\cos\theta$

Now $\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \times \frac{d\theta}{dx}$

$= \frac{d}{d\theta}(-\frac{1}{2}\cos\theta) \times \frac{d\theta}{dx}$

$= \frac{1}{2}\sin\theta \times \frac{1}{(-4\sin\theta)}$

$= -\frac{1}{8}$

Example 6

Find the coordinates of the stationary points on the curve defined in terms of a parameter t by $x = t^2 + 1$ and $y = t^3 - 12t$. Use the second derivative to determine the nature of these points.

Solution

$\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2 - 12$

$\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right)$

$= \frac{3t^2 - 12}{2t}$

Graphical calculator support pack

At stationary points on the curve $\frac{dy}{dx} = 0$.

So $\frac{3t^2 - 12}{2t} = 0$

$\Rightarrow \quad 3t^2 = 12$

$\Rightarrow \quad t^2 = 4$

$\Rightarrow \quad t = 2$ or $t = -2$

Solve this equation by equating the numerator to zero.

Solution

$\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 4 - 2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= \frac{4 - 2t}{2t}$$

At stationary points on the curve, $\frac{dy}{dx} = 0$.

$$\frac{4 - 2t}{2t} = 0 \quad \Rightarrow \quad 4 - 2t = 0$$
$$t = 2$$

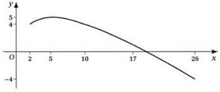
So the stationary point occurs at (5, 5).
To determine its nature we need to evaluate the second derivative.

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$
$$= \frac{d}{dt}\left(\frac{4 - 2t}{2t}\right) \times \frac{dt}{dx}$$
$$= -\frac{8}{4t^2} \times \frac{1}{2t}$$
$$= -\frac{1}{t^3}$$

When $t = 2$, $\frac{d^2y}{dx^2} < 0$, so (5, 5) is a maximum point.
To sketch the curve we can evaluate x- and y-coordinates for different values of the parameter.

t	1	2	3	4	5
x	2	5	10	17	26
y	4	5	4	1	−4

Together with what we know about the stationary points, we can produce a sketch of the curve:



11.6 Parametric Differentiation

Exercise

Technique

1 Eliminate the parameter to find the Cartesian equations of the following curves:

1 a

a $x = 3t, y = t^2$

b $x = 2t, y = t^3 + 1$

c $x = 5\cos t, y = 5\sin t$

d $x = 4\cos\theta, y = 3\sin\theta$

e $x = 1 + 2\cos\theta, y = 3 + 4\cos\theta$

f $x = a\cos\theta, y = b\sin\theta$

2 Find an expression for $\frac{dy}{dx}$ for each of the following parametrically defined curves:

a $x = 1 - t^2$
$y = 6t + 5$

b $x = \cos^2\theta$
$y = 1 - \sin\theta$

c $x = \frac{1}{t+1}$
$y = \frac{1}{t-1}$

3 Find the coordinates of the points on each of the following curves where the gradient has the value indicated:

a $x = t^3 - 3t$
$y = 2t + 1$
$\frac{dy}{dx} = \frac{2}{9}$

b $x = \frac{6}{t}$
$y = t^2$
$\frac{dy}{dx} = -9$

c $x = \tan\theta$
$y = 2 + \sin\theta$
$\frac{dy}{dx} = \frac{1}{8}$

4 Find the equations of the tangent and the normal to the curve with parametric equations $x = e^t, y = 1 + e^{2t}$, at the point where $t = 1$. 4

5 The curve C has parametric equations $x = t^4$ and $y = 2t - 1$. Find a general equation for the tangent at some point $(t^4, 2t - 1)$ on the curve. Hence, write down the equation of the tangent to the curve at (16, 3).

6 Find the coordinates of the stationary points on each of the following parametrically defined curves. Find also an expression for the second derivative and use it to determine the nature of these stationary points. Hence, or otherwise, sketch the curve.

a $x = 3t + 1$
$y = t^3 - 6t^2$

b $x = t^3 + 1$
$y = t^2 + t$

So the linear approximation to the function gives a value for the definite integral that is correct to two decimal places. This approach is particularly useful when evaluating integrals that are difficult to evaluate exactly.

The binomial theorem can only be used to find the series expansions of functions of the form $(a+x)^n$, where n is a rational power. Functions such as $\sin x$, $\cos x$, e^x and $\ln x$ can also be expressed as a series of ascending powers of x using the Maclaurin series.

The Maclaurin series

Suppose $f(x)$ is some function of x that can be written as a series of ascending powers of x, such that

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_rx^r + \ldots$$

The Maclaurin series expansion is named after Colin Maclaurin (1698–174

where a_r is the coefficient of the x^r term. Assume that $f(x)$ is a function that can be continuously differentiated to find $f'(x)$, $f''(x)$ and its higher derivatives. It follows that

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$$

$$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \ldots$$

$$f'''(x) = 6a_3 + 24a_4x + 60a_5x^2 + 120a_6x^3 + \ldots$$

$$f^{(4)}(x) = 24a_4 + 120a_5x + 360a_6x^2 + 840a_7x^3 + \ldots \text{ and so on.}$$

$f^{(4)}(x)$ denotes the fourth derivative of $f($

Substituting $x=0$ into these expansions,

$$f(0) = a_0 \quad \Rightarrow \quad a_0 = f(0)$$

$$f'(0) = a_1 \quad \Rightarrow \quad a_1 = f'(0)$$

$$f''(0) = 2a_2 = 2! \times a_2 \quad \Rightarrow \quad a_2 = \frac{f''(0)}{2!}$$

$$f'''(0) = 6a_3 = 3! \times a_3 \quad \Rightarrow \quad a_3 = \frac{f'''(0)}{3!}$$

$$f^{(4)}(0) = 24a_4 = 4! \times a_4 \quad \Rightarrow \quad a_4 = \frac{f^{(4)}(0)}{4!}$$

More generally, $\qquad a_r = \dfrac{f^{(r)}(0)}{r!}$

where $f^{(r)}(0)$ is the value of the rth derivative of $f(x)$ at $x=0$. Substituting these values for the coefficients $a_0, a_1, a_2, a_3, a_4, \ldots, a_r, \ldots$ into the series for $f(x)$ gives

$$f(x) = f(0) + xf'(0) + x^2\frac{f''(0)}{2!} + x^3\frac{f'''(0)}{3!} + \ldots + x^r\frac{f^{(r)}(0)}{r!} + \ldots$$

◀ **Learn this result.**

Example 5

Use the Maclaurin series expansion to find the cubic approximation of $\ln(1+x)$. Hence find the approximate value of $\ln(1.1)$.

Solution

$$f(x) = \ln(1+x) \quad \Rightarrow \quad f(0) = \ln 1 = 0$$
$$f'(x) = (1+x)^{-1} \quad \Rightarrow \quad f'(0) = 1$$
$$f''(x) = -(1+x)^{-2} \Rightarrow f''(0) = -1$$
$$f'''(x) = 2(1+x)^{-3} \Rightarrow f'''(0) = 2, \text{ and so on.}$$

Recall that $f'(x) = \frac{1}{(1+x)} = (1+x)$

The Maclaurin series for $\ln(1+x)$ is therefore

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \dots$$

and the cubic approximation is

$$\ln(1+x) \approx x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3$$

Substituting $x = 0.1$ gives

$$\begin{aligned}\ln(1.1) &\approx 0.1 - \tfrac{1}{2}(0.1)^2 + \tfrac{1}{3}(0.1)^3 \\ &\approx 0.1 - 0.005 + 0.000\,3\dot{3} \\ &\approx 0.0953 \text{ (4 d.p.)}\end{aligned}$$

Check this result usi calculator.

By evaluating the higher derivatives of $\ln(1+x)$ when $x = 0$, it is possible to show that the Maclaurin series for $\ln(1+x)$ is

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \dots$$

This expansion is only valid for $-1 < x \leq 1$.

Notice the condition the inequalities.

Replacing x with $-x$ in this series gives

$$\ln(1-x) = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5 - \dots$$

This expansion is only valid for $-1 \leq x < 1$.

The Maclaurin expansion cannot be used to find a power series for $\ln x$ because the function and its derivatives $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$, and so on, are not defined for $x = 0$.

11.7 Maclaurin Series
Exercise
Technique

1 Find the first three non-zero terms in the Maclaurin series expansions for:

a $\cos\frac{1}{4}x$ **b** x^2e^{5x} **c** $\ln(1+3x)$
d $\tan x$ **e** $\tan^{-1}x$ **f** $e^x\cos x$

2 Use the quadratic approximation for $\cos 2x$ obtained from its Maclaurin series to solve the equation $\cos 2x = 3x$, correct to two decimal places.

3 Find the first three non-zero terms in the Maclaurin series expansion of $\ln(1+2x)$ and $\ln(1-2x)$. Hence, write down a cubic approximation for $\ln\left(\frac{1+2x}{1-2x}\right)$ and, by substituting $x = 0.1$ into it, find the approximate value of $\ln 1.5$ to three decimal places.

Consolidation

Exercise A

1 Given that $y = xe^{-3x}$, find $\frac{dy}{dx}$. Hence find the coordinates of the stationary point on the curve $y = xe^{-3x}$.

(UCLES)

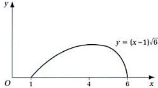
2 A curve is defined by the parametric equations $x = t^2 + 3t$ and $y = 2t - t^2$.

a Find $\frac{dy}{dx}$ in terms of t.

b The normal to the curve at the point P has gradient $\frac{3}{8}$. Determine the coordinates of P.

(AEB)

3



The diagram shows a rough sketch of the graph of $y = (x-1)\sqrt{6-x}$.

a Find an expression for $\frac{dy}{dx}$.

b By considering the gradient of the curve at the point where $x = 4$, determine whether the x-coordinate of the maximum point on the graph is less than or greater than 4.

(NEAB)

4 A curve C is given by the equations $x = 2\cos t + \sin 2t$, $y = \cos t - 2\sin 2t$, $0 \le t \le \pi$, where t is a parameter.

a Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in terms of t.

b Find the value of $\frac{dy}{dx}$ at the point P on C, where $t = \frac{\pi}{4}$.

c Find an equation of the normal to the curve at P.

(ULEAC)

5 Use Maclaurin's theorem to show that for sufficiently small values of x,

$$(1 + x^2)\tan^{-1} x \approx x + \tfrac{2}{3}x^3 - \tfrac{2}{15}x^5$$

(NICCEA)

6 A curve is defined implicitly by the equation $x^2y + y^2 - 3x - 3 = 0$. Point A has coordinates (1, 2) and point B is where the curve crosses the x-axis.

a Show that point A lies on the curve.
b Find the coordinates of point B.
c Calculate the gradient of the curve at point A.
d Find the equation of the normal to the curve at point A.

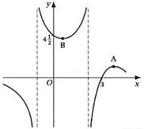
(AEB)

7 The parametric equations of a curve are $x = e^{2t} - 5t$ and $y = e^{2t} - 2t$. Find $\frac{dy}{dx}$ in terms of t. Find the exact value of t at the point on the curve where the gradient is 2.

(UCLES)

8 The sketch shows the curve with equation $y = \frac{3x-9}{x^2-x-2}$.

a Write down the equations of the vertical asymptotes.
b Find $\frac{dy}{dx}$ in terms of x and hence determine the coordinates of the maximum and minimum points, A and B, on the curve.



(AEB)

9 It is given that $y = \frac{1}{1+\sin 2x}$. Show that when $x = 0$, $\frac{d^2y}{dx^2} = 8$. Find the first three terms in Maclaurin's series for y.

a Use the series to obtain an approximate value for $\int_{-0.1}^{0.1} y\,dx$, giving your answer correct to four decimal places.
b Find the first two terms of Maclaurin's series for $\frac{dy}{dx}$.

(OCSEB)

10 An oil rig, O, is situated out at sea at a perpendicular distance of 10 km from a straight coastline. The point on the coast nearest to O is P. A refinery, R, is situated on the coast 10 km from P. A project is being planned to bring the oil ashore by means of pipelines running from O directly to a point Q on the coastline and then along the coast to R. The project will cost £5 million per kilometre under the sea and £3 million per kilometre along the coast. Show that if Q is x km from P, the cost of the project will be £C million, where $C = 5\sqrt{100 + x^2} + 30 - 3x$. Show that $\frac{dC}{dx} = 0$ when $x = 7.5$. Find $\frac{d^2C}{dx^2}$ and hence, or otherwise, determine whether C is a maximum or minimum when $x = 7.5$.

(NEAB)

Exercise B

1 A curve has equation $y = x^2 - \frac{54}{x}$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x. Calculate the coordinates of the stationary point of the curve and determine whether it is a maximum or minimum point.

(AEB)

2 **a** Use Maclaurin's theorem to derive the series expansion for $\cos x$, giving the first three non-zero terms.

b Hence obtain the first three non-zero terms of the series expansions for

i $\cos 2x$ **ii** $\cos^2 x$

(NICCEA)

3 A curve C is defined by the parametric equations $x = 4t + \frac{4}{t}$ and $y = 4t - \frac{4}{t}$.

a Express $\frac{dy}{dx}$ in terms of t, simplifying your answer.

b At the point on the curve C where $t = 2$:

i show that $\frac{dy}{dx} = \frac{5}{3}$

ii find the equation of the normal to the curve.

(NEAB)

4 The number of bacteria present in a culture at time t hours after the beginning of an experiment is denoted by N. The relation between N and t is modelled by $N = 100e^{3t/2}$.

a After how many hours will the number of bacteria be 9000?

b At what rate per hour will the number of bacteria be increasing when $t = 6$?

(UCLES)

5 A curve is given parametrically by the equations $x = 2\theta + \sin 2\theta$, $y = 2\cos^2\theta$, $0 \le \theta \le \frac{\pi}{2}$. Show that $\frac{dy}{dx} = -\tan\theta$ (given $\theta \ne \frac{\pi}{2}$). Find the equation of the tangent to the curve at the point where $\theta = \frac{\pi}{4}$.

(AEB)

6 Given that $f(x) = \frac{4x+3a}{x^2+a^2}$, show that $f'(2) = \frac{4(a^2-3a-4)}{(4+a^2)^2}$. Find the two values of the constant a for which $f(x)$ has a stationary value when $x = 2$.

(OCSEB)

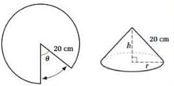
7 A curve has equation $x^3 + y^3 + 2x + 5y = 9$. Find an expression for $\frac{dy}{dx}$ in terms of x and y. Hence show that the gradient of the curve is never positive.

(OCSEB)

Applications and Activities

1 *Making a cone*

Cut a sector of angle θ from a circular piece of card of radius 20 cm and attach the two straight edges together to make a right-circular cone of height h cm and base radius r cm.



a Find an expression for the perpendicular height, h, of the cone in terms of its base radius, r.

b Use this to find an expression for the volume, V, of the cone in terms of r only.

c Use the product rule to differentiate V with respect to r. Find the exact value of r for which $\frac{dV}{dr} = 0$, and the corresponding value of V. Show that this is the maximum possible volume for a cone made from this particular piece of card.

d Find an expression, in terms of θ, for the length of the circular edge of the piece of card from which the cone is made. Equating this to the circumference of the base of the cone, find the value of θ that corresponds to the maximum possible volume of the cone.

e Repeat these calculations for a cone made from a circular piece of card of radius R cm. In what way is the angle θ that corresponds to the maximum possible volume of the cone dependent on the radius R?

Summary

- The second derivative $\frac{d^2y}{dx^2}$ is used to determine the nature of stationary points:

$$\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{d^2y}{dx^2} > 0 \Rightarrow \text{minimum point}$$

$$\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{d^2y}{dx^2} < 0 \Rightarrow \text{maximum point}$$

- When $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, the gradient must be considered on either side of the stationary point to determine whether it is a maximum point, a minimum point or a point of inflexion.

- A **non-stationary point of inflexion** is a point where the gradient of the curve is at its maximum or minimum value locally. The tangent to the curve at these points crosses from one side of the curve to the other.

- Types of inflexion can be determined by considering the first and third derivatives:

$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	Type of inflexion
+ve	0	+ve	
+ve	0	−ve	
−ve	0	−ve	
−ve	0	+ve	

- The **product rule** for differentiation is:

$$\text{If } y = uv, \text{ then } \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = uv' + vu'$$

- The **quotient rule** for differentiation is:

$$\text{If } y = \frac{u}{v}, \text{ then } \frac{dy}{dx} = \frac{\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)}{v^2} = \frac{(vu' - uv')}{v^2}$$

- $\frac{d}{dx}(e^x) = e^x$ $\quad$ $\frac{d}{dx}(a^x) = a^x \ln a$

- $\frac{d}{dx}(\ln x) = \frac{1}{x}$ $\quad$ $\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$

- $\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad$ $\frac{d}{dx}(\cot x) = \operatorname{cosec}^2 x$

  $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ $\quad$ $\frac{d}{dx}(\sec x) = \sec x \tan x$

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$ $\quad$ $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$

  $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$

- A function in x and y that cannot be written in the form $y = f(x)$ is an **implicitly defined function**. The method used to find the gradient of an implicit function is **implicit differentiation**, and relies on the chain rule. The product rule and/or the quotient rule may also be needed.

- The first and second derivatives, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, can be found for parametrically defined curves using the chain rule.

- Maclaurin's theorem can be used to find series expansions for standard functions. For values of x close to 0:

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \ldots + \frac{x^r f^{(r)}(0)}{r!} + \ldots$$

where $f^{(r)}(0)$ is the rth derivative of $f(x)$ evaluated at $x = 0$.

12 Integration II

What you need to know

- How to integrate polynomials and basic trigonometrical functions by reversing differentiation.
- How to use integration to calculate areas and volumes.
- How to differentiate standard functions and use the product rule.
- How to manipulate algebraic fractions.
- How to express rational functions as partial fractions.
- How to differentiate implicit functions.

Review

1 Integrate the following with respect to x:

a $x^3 + 5x^2 - x + 10$ **b** $(x-4)^2$ **c** $\sin x$
d $\cos 3x$ **e** x^n **f** $\sin kx$

Remember that integration is the reve of differentiation.

2 Sketch the curve $y = x^2 + 1$.

a Find the area under the curve between $x = 0$ and $x = 1$.
b Find the volume generated when this area is rotated about the x-axis through one complete turn.

3 Differentiate the following functions with respect to x:

a $\sin(4x+1)$ **e** $\ln x$ **h** $\ln(x^2+7)$
b e^x **f** $\ln 3x$ **i** xe^x
c e^{kx} **g** $\ln(2x+13)$ **j** $x \sin 2x$
d e^{1+3x}

4 Express each of the following as a proper fraction:

a $\dfrac{x+1}{x-1}$ **b** $\dfrac{x}{x+1}$

5 Express each of the following as partial fractions:

a $\dfrac{x-7}{(x+1)(x+3)}$ **b** $\dfrac{3x-1}{(x+1)(x^2+1)}$ **c** $\dfrac{2x^2-7x-18}{x^2-7x+10}$

6 Find $\frac{dy}{dx}$ for each of the following using implicit differentiation:

a $y^2 = x+1$ **b** $y^2 = x^2 + 2x - 3$

Example 1

Find:

a $\int \sin 5x\,dx$ **b** $\int \sin(4-2x)\,dx$
c $\int_0^1 \cos(3-x)\,dx$ **d** $\int_0^{\pi/8} \sec^2 2x\,dx$

Solution

a $\int \sin 5x\,dx = -\frac{1}{5}\cos 5x + c$ ◀ $\int \sin kx\,dx = -\frac{1}{k}\cos kx + c$

Check that differentiation revers[es] the integration proce[ss]

b $$\int \sin(4-2x)\,dx = -\frac{1}{(-2)}\cos(4-2x) + c$$
$$= \frac{1}{2}\cos(4-2x) + c$$

c $$\int_0^1 \cos(3-x)\,dx = \left[\frac{1}{-1}\sin(3-x)\right]_0^1$$
$$= (-\sin 2) - (-\sin 3)$$
$$= -\sin 2 + \sin 3$$
$$= -0.768 \quad (3\text{ d.p.})$$

Use radians when integrating.

Use a calculator to check that $-\sin 2 + \sin 3 = -0.$

d $$\int_0^{\pi/8} \sec^2 2x\,dx = \left[\tfrac{1}{2}\tan 2x\right]_0^{\pi/8}$$
$$= \left(\tfrac{1}{2}\tan\tfrac{\pi}{4}\right) - \left(\tfrac{1}{2}\tan 0\right)$$
$$= \left(\tfrac{1}{2}\times 1\right) - \left(\tfrac{1}{2}\times 0\right) = \tfrac{1}{2}$$

$\frac{d}{dx}(\tan kx) = k\sec^2 kx$
$\int \sec^2 kx\,dx = \frac{1}{k}\tan kx$

Remember to use double angle formulas when integrating $\sin^2 x$ or $\cos^2 x$ (see Chapter 9). Sometimes the compound angle formulas can be used to transform an integral into a 'simpler form'.

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \quad [1]$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B \quad [2]$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \quad [3]$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B \quad [4]$$

Example 2

Find:

a $\int \sin 5x\cos 2x\,dx$
b $\int_0^{\pi/4} \sin x\sin 3x\,dx$

Solution

a Adding formulas [1] and [2],

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$

Now let $A = 5x$ and $B = 2x$

$$\begin{aligned} \text{Then}\quad \sin(5x+2x) + \sin(5x-2x) &= 2\sin 5x\cos 2x \\ \Rightarrow\quad \sin 7x + \sin 3x &= 2\sin 5x\cos 2x \\ \Rightarrow\quad \tfrac{1}{2}(\sin 7x + \sin 3x) &= \sin 5x\cos 2x \end{aligned}$$

The product $\sin 5x\cos 2x$ has now been transformed into the sum of two trigonometrical functions, which can be integrated.

$$\begin{aligned} \int \sin 5x\cos 2x\,dx &= \int \tfrac{1}{2}(\sin 7x + \sin 3x)\,dx \\ &= \tfrac{1}{2}\int(\sin 7x + \sin 3x)\,dx \\ &= \tfrac{1}{2}\left[-\tfrac{1}{7}\cos 7x - \tfrac{1}{3}\cos 3x\right] + c \\ &= -\tfrac{1}{14}\cos 7x - \tfrac{1}{6}\cos 3x + c \end{aligned}$$

A constant multiplier can be moved outside the integration symbol.

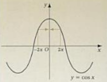
b Subtracting formula [4] from formula [3],

$$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$$

Let $A = x$ and $B = 3x$. Then

$$\begin{aligned} \cos(x+3x) - \cos(x-3x) &= -2\sin x\sin 3x \\ -\tfrac{1}{2}[\cos 4x - \cos(-2x)] &= \sin x\sin 3x \end{aligned}$$

Recall that cosine is an even function, so that $\cos(-2x) = \cos 2x$.

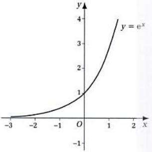


So $\quad \sin x\sin 3x = -\tfrac{1}{2}(\cos 4x - \cos 2x)$

$$\begin{aligned} \text{Now}\quad \int_0^{\pi/4} \sin x\sin 3x\,dx &= \int_0^{\pi/4} -\tfrac{1}{2}(\cos 4x - \cos 2x)\,dx \\ &= -\tfrac{1}{2}\int_0^{\pi/4}(\cos 4x - \cos 2x)\,dx \\ &= -\tfrac{1}{2}\left[\tfrac{1}{4}\sin 4x - \tfrac{1}{2}\sin 2x\right]_0^{\pi/4} \\ &= -\tfrac{1}{2}\left[\left(\tfrac{1}{4}\sin\pi - \tfrac{1}{2}\sin\tfrac{\pi}{2}\right) - \left(\tfrac{1}{4}\sin 0 - \tfrac{1}{2}\sin 0\right)\right] \\ &= -\tfrac{1}{2}\left(\tfrac{1}{4}\times 0 - \tfrac{1}{2}\times 1 - 0 + 0\right) \\ &= -\tfrac{1}{2}\left(-\tfrac{1}{2}\right) = \tfrac{1}{4} \end{aligned}$$

The exponential function

Remember that one important feature of the exponential function is that its gradient function is equal to the function itself. By applying the chain rule to the exponential function, derivatives can be found as follows, and by thinking of integration as the reverse of differentiation we can write down another set of standard integrals.



$$\frac{d}{dx}(e^x) = e^x \quad \Rightarrow \quad \boxed{\int e^x\,dx = e^x + c} \qquad [1]$$

◄ **Learn these important results.**

$$\frac{d}{dx}(e^{ax}) = ae^{ax} \quad \Rightarrow \quad \boxed{\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c} \qquad [2]$$

$$\frac{d}{dx}(e^{ax+b}) = ae^{ax+b} \quad \Rightarrow \quad \boxed{\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c} \qquad [3]$$

Example 3

Find:

a $\int e^{-x}\,dx$ **b** $\int e^{2x}\,dx$ **c** $\int e^{\frac{1}{2}x}\,dx$ **d** $\int_0^1 e^{2-5x}\,dx$

Solution

a $\displaystyle\int e^{-x}\,dx = \frac{1}{-1}e^{-x} + c$ ◄ **Using result [2], with $a = -1$.**

$= -e^{-x} + c$

b $\displaystyle\int e^{2x}\,dx = \tfrac{1}{2}e^{2x} + c$ ◄ **Using result [2], with $a = 2$.**

c $\displaystyle\int e^{\frac{1}{2}x}\,dx = \frac{1}{\left(\frac{1}{2}\right)}e^{\frac{1}{2}x} + c = 2e^{\frac{1}{2}x} + c$

d $\displaystyle\int_0^1 e^{2-5x}\,dx = \left[-\tfrac{1}{5}e^{2-5x}\right]_0^1$

$= \left(-\tfrac{1}{5}e^{2-5}\right) - \left(-\tfrac{1}{5}e^{2-0}\right)$

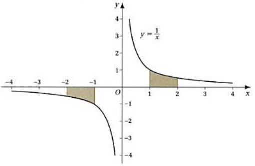
$= -\tfrac{1}{5}e^{-3} + \tfrac{1}{5}e^{2}$

$= \tfrac{1}{5}(e^2 - e^{-3}) = 1.468$ (3 d.p.)

It is always useful to check your integratic by mentally differentiating your answer. In **b**, $\frac{d}{dx}\left(\frac{1}{2}e^{2x} + c\right) = \frac{1}{2} \times 2e$

Recall that $\dfrac{1}{\left(\frac{1}{2}\right)} = 2$.

Definite integrals involving logarithms



The diagram shows the graph of $y = \frac{1}{x}$. The areas of the shaded regions are equal. Since we find the area under the curve by integrating between limits, we would expect $\int_1^2 \frac{1}{x}\,dx = -\int_{-2}^{-1} \frac{1}{x}\,dx$.

The minus sign indic that the area is below x-axis.

$$\int_1^2 \frac{1}{x}\,dx = [\ln x]_1^2$$
$$= \ln 2 - \ln 1$$
$$= \ln 2$$

$$\int_{-2}^{-1} \frac{1}{x}\,dx = [\ln x]_{-2}^{-1}$$
$$= \ln(-1) - \ln(-2)$$

But we get stuck here because we cannot find the logarithm of a negative number on a calculator; the logarithmic function is only defined for positive values of x. To avoid unnecessary complications with logarithms of negative numbers, we use the modulus function when integrating with the logarithmic function.
In this example,

$$\int_{-2}^{-1} \frac{1}{x}\,dx = [\ln|x|]_{-2}^{-1}$$
$$= \ln|-1| - \ln|-2|$$
$$= \ln 1 - \ln 2$$
$$= -\ln 2$$

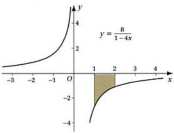
Remember that the modulus function tel us to take the absolu value.

In general:

$$\int_a^b \frac{f'(x)}{f(x)}\,dx = [\ln|f(x)|]_a^b \quad \text{and} \quad \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

Check Example 4. N that the modulus function is needed fo parts but **d**, because possible for $f(x) < 0$.

Example 5



The diagram shows part of the curve $y = \frac{8}{1-4x}$. Find the area of the shaded region.

Solution

$$\begin{aligned}
\text{shaded area} &= \int_1^2 \frac{8}{1-4x}\,\mathrm{d}x \\
&= -2\int_1^2 \frac{-4}{1-4x}\,\mathrm{d}x \\
&= -2[\ln|1-4x|]_1^2 \\
&= -2[\ln|-7| - \ln|-3|] \\
&= -2[\ln 7 - \ln 3] \\
&= -2\ln\left(\tfrac{7}{3}\right)
\end{aligned}$$

Remember to take the limits from the diagram.

Use the modulus form because $1 - 4x$ is negative for $x > \frac{1}{4}$.

The minus sign signifies that the area is below the x-axis, so the shaded area is $2\ln\left(\frac{7}{3}\right)$.

Inverse trigonometric functions

Another set of important integrals can be found by studying the results of differentiating inverse trigonometrical functions. Begin with $\sin^{-1}\left(\frac{x}{a}\right)$ and $\tan^{-1}\left(\frac{x}{a}\right)$, where a is a constant. Using general functions like these will allow us to establish another set of general integrals.

Recall the alternative notation $\arcsin\left(\frac{x}{a}\right)$ and $\arctan\left(\frac{x}{a}\right)$.

Let $\quad y = \sin^{-1}\left(\dfrac{x}{a}\right)$

Then $\quad \sin y = \dfrac{x}{a}$

Recall from Chapter 11 how to differentiate $\sin y$ implicitly with respect to x.

Differentiating with respect to x,

$$\cos y \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a}$$

So $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{a\cos y}$

$$\frac{dy}{dx} = \frac{1}{a \times \left(\frac{a^2 + x^2}{a^2}\right)}$$

$$= \frac{1}{\left(\frac{a^2 + x^2}{a}\right)} = \frac{a}{a^2 + x^2}$$

So $\int \frac{a}{a^2 + x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c$ ◀ **Learn this result.**

and $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

Remember that the constant a can be taken outside the integration symbol.

Example 6

Find:

a $\int \frac{1}{\sqrt{1 - x^2}}\,dx$ **b** $\int \frac{1}{2 + x^2}\,dx$

Solution

a Using the standard result $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$, with $a = 1$

$$\int \frac{1}{\sqrt{1 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{1}\right) + c = \sin^{-1} x + c$$

b Using the standard result $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$, with $a = \sqrt{2}$

$$\int \frac{1}{2 + x^2}\,dx = \int \frac{1}{(\sqrt{2})^2 + x^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$$

Example 7

Evaluate $\int_0^1 \frac{3}{\sqrt{4 - x^2}}\,dx$.

Solution

Rearrange the integral in standard form by extracting a factor of 3 outside the integral sign and changing the 4 to 2^2.

$$\int_0^1 \frac{3}{\sqrt{4 - x^2}}\,dx = 3\int_0^1 \frac{1}{\sqrt{2^2 - x^2}}\,dx$$

$$= 3\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1$$

$$= 3[\sin^{-1}(\tfrac{1}{2}) - \sin^{-1}(0)]$$

$$= 3\left[\frac{\pi}{6} - 0\right] = \frac{\pi}{2}$$

Remember to work in radians.

Recall the special angles.

12.1 Standard Integrals
Exercise
Technique

1 Find:

a $\int \sin 3x \, dx$ b $\int \cos 4x \, dx$ c $\int \sin(2x-1) \, dx$
d $\int \cos(1+2x) \, dx$ e $\int \sec^2 3x \, dx$ f $\int (1+\sec^2 2x) \, dx$

1 a

2 Evaluate:

a $\int_0^{\pi/4} \sin 2\theta \, d\theta$ b $\int_0^{3/2} \cos(1+2\theta) \, d\theta$
c $\int_0^{\pi/4} \sec^2 x \, dx$ d $\int_0^{\pi/6} \sec^2 2x \, dx$

3 Find:

a $\int \sin 2x \cos x \, dx$ b $\int \cos 2x \cos x \, dx$
c $\int_0^{\pi/2} \sin 3x \sin x \, dx$ d $\int_0^{\pi} \sin x \cos 2x \, dx$

4 Find:

a $\int e^{5x} \, dx$ d $\int e^{\frac{1}{3}x} \, dx$ g $\int 2e^{2x} \, dx$
b $\int e^{-x} \, dx$ e $\int e^{\frac{1}{2}x} \, dx$ h $\int \frac{1}{2} e^{-6x} \, dx$
c $\int \frac{1}{e^{3x}} \, dx$ f $\int e^{-\frac{1}{4}x} \, dx$ i $\int 3e^{-0.1x} \, dx$

4 f

5 Find:

a $\int (2x + 3e^{2x}) \, dx$ d $\int (1 - e^{1-2x}) \, dx$ f $\int_0^2 (x - 3e^{-x}) \, dx$
b $\int (1 - e^{-x}) \, dx$ e $\int_0^1 2e^{4x} \, dx$ g $\int_0^{1/2} e^{1+2x} \, dx$
c $\int (1 + e^x)^2 \, dx$

6 Find:

a $\int \frac{4}{x} \, dx$ b $\int \frac{1}{2x} \, dx$ c $\int \frac{x+1}{x} \, dx$ d $\int \frac{1}{x+2} \, dx$
e $\int \frac{4}{2x+1} \, dx$ f $\int \frac{3}{7+6x} \, dx$ g $\int \frac{5}{2-3x} \, dx$ h $\int \frac{1}{1-7x} \, dx$

6 b

7 Find:

a $\int \frac{2x}{1+x^2} \, dx$ b $\int \frac{x}{4+3x^2} \, dx$ c $\int \frac{2x+1}{x^2+x} \, dx$
d $\int \frac{x}{1-x^2} \, dx$ e $\int_1^2 \frac{1}{2x-1} \, dx$ f $\int_0^1 \frac{x}{1+x^2} \, dx$

7 b

8 Find:

a $\int \cot x \, dx$ b $\int \frac{\sin x}{1+\cos x} \, dx$ c $\int \frac{\cos x}{1+\sin x} \, dx$

9 Find:

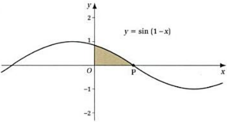
a $\int \frac{1}{\sqrt{1-x^2}} \, dx$ b $\int \frac{1}{\sqrt{9-x^2}} \, dx$ c $\int \frac{1}{9+x^2} \, dx$
d $\int \frac{2}{3+x^2} \, dx$ e $\int \frac{4}{\sqrt{2-x^2}} \, dx$ f $\int \frac{1}{7+x^2} \, dx$

9 a

Contextual

1 Sketch the curve $y = 1 + \sin 2x$ for $0 \le x \le \pi$. Find the area bound by the curve, the y-axis and the positive x-axis, leaving your answer in terms of π.

2

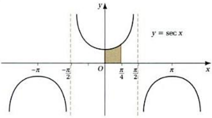


The graph shows the curve $y = \sin(1 - x)$

a Verify that the x-coordinate of P is 1.

b Calculate the area of the shaded region, giving your answer correct to two decimal places.

3



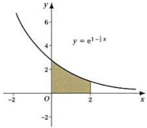
The diagram shows part of the graph of $y = \sec x$. Find the volume generated when the shaded area is rotated about the x-axis through four right-angles.

4 **a** Use $1 + \tan^2 A \equiv \sec^2 A$ to express $\tan^2 2x$ in terms of $\sec^2 2x$.

b Find $\int \tan^2 2x \, dx$.

5 If $g'(x) = 12\cos(1 + 2x)$ and $g(-\frac{1}{2}) = 1$, find $g(x)$.

6 The diagram shows the graph of $y = e^{1-\frac{1}{2}x}$. Calculate the area of the shaded region, leaving your answer in terms of e.

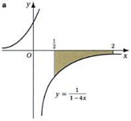


7 Sketch the graph of $y = \frac{1}{2}e^{2x}$. Find the volume generated when the area under the curve between $x = 0$ and $x = 1$ is rotated about the x-axis through one complete revolution.

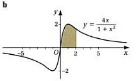
8 If $2^x = e^m$ find an expression for m in terms of x. Hence evaluate $\int_0^1 2^x \, dx$.

9 Sketch the graph of $y = \frac{2}{x}$. Calculate the area under the graph between $x = 1$ and $x = 4$.

10 **a**

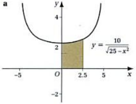


b

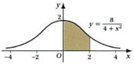


Calculate the areas of the shaded regions.

11 **a**



b



Calculate the areas of the shaded regions.

12.2 Rational Functions

A **rational function** is a fraction where both numerator and denominator are polynomials (the numerator can be a constant). Some examples are:

$$\frac{x}{1+x^2} \quad \frac{x-1}{x+1} \quad \frac{x^2}{1+x} \quad \frac{2x-1}{(x+1)(x-2)}$$

Recall that when the degree of the numerator is less than the degree of the denominator it is called a **proper fraction**. When the degree of the numerator is equal to or greater than the degree of the denominator, it is called an **improper fraction**.

When integrating rational functions, first check to see whether the fraction is proper or improper. For example, $\frac{x}{1+x^2}$ is a proper fraction and can be written in the form $\frac{kf'(x)}{f(x)}$, where k is a constant.

$$\text{So} \quad \int \frac{x}{1+x^2}\,dx = \tfrac{1}{2}\int \frac{2x}{1+x^2}\,dx \qquad \blacktriangleleft \textbf{ Recall } \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c.$$

$$= \tfrac{1}{2}\ln(1+x^2) + c$$

The modulus function isn't required because $1+x^2 > 0$ for all x.

Improper fractions cannot be written in the form $\frac{kf'(x)}{f(x)}$, where k is a constant. To integrate an improper fraction first write it as the sum or difference of proper algebraic fractions. Then try to use the list of standard integrals.

Simple improper fractions

All improper fractions can be expressed as the sum or difference of a polynomial and a proper fraction.

One way of doing this is to write the numerator in terms of the denominator.

Example 1

Find:

a $\displaystyle\int \frac{x}{1-x}\,dx$ **b** $\displaystyle\int \frac{x+1}{x-1}\,dx$ **c** $\displaystyle\int \frac{x^3}{7+x^2}\,dx$

Solution

a First express $\frac{x}{1-x}$ as the sum of a polynomial and a proper fraction. Do this by writing the numerator in terms of the denominator

$$\frac{x}{1-x} = \frac{-(1-x)+1}{1-x} = \frac{-(1-x)}{1-x} + \frac{1}{1-x}$$

$$= -1 + \frac{1}{1-x}$$

Now $\int \frac{x}{1-x}\,dx = \int\left(-1+\frac{1}{1-x}\right)dx$

$= -x + \int \frac{1}{1-x}\,dx$

$= -x - \int \frac{-1}{1-x}\,dx$ ◄ **Use** $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c.$

$= -x - \ln|1-x| + c$

The integral is now in the standard form $\frac{f'(x)}{f(x)}$ where $f(x) = 1 - x$.

Check this result by differentiation.

b First write $\frac{x+1}{x-1}$ as a sum involving a proper fraction.

$$\frac{x+1}{x-1} = \frac{(x-1)+2}{x-1}$$

$$= \frac{x-1}{x-1} + \frac{2}{x-1}$$

$$= 1 + \frac{2}{x-1}$$

Check that $x - 1 + 2 = x + 1$.

Now $\int \frac{x+1}{x-1}\,dx = \int\left(1+\frac{2}{x-1}\right)dx$

$= x + 2\int \frac{1}{x-1}\,dx$ ◄ **The form** $\int \frac{f'(x)}{f(x)}\,dx.$

$= x + 2\ln|x-1| + c$

Integrate each term separately.

Check this result by differentiation.

c $$\frac{x^3}{7+x^2} = \frac{7x+x^3-7x}{7+x^2} = \frac{x(7+x^2)-7x}{7+x^2}$$

$$= \frac{x(7+x^2)}{7+x^2} - \frac{7x}{7+x^2}$$

$$= x - \frac{7x}{7+x^2}$$

Write as a proper algebraic fraction first.

Now $\int \frac{x^3}{7+x^2}\,dx = \int\left(x - \frac{7x}{7+x^2}\right)dx$

$= \frac{1}{2}x^2 - 7\int \frac{x}{7+x^2}\,dx$

$= \frac{1}{2}x^2 - \frac{7}{2}\int \frac{2x}{7+x^2}\,dx$ ◄ **The form** $\int \frac{f'(x)}{f(x)}\,dx.$

$= \frac{1}{2}x^2 - \frac{7}{2}\ln(7+x^2) + c$

Integrate each term separately.

Check this result by differentiation.

Partial fractions

If the denominator can be factorised, partial fractions can be used to simplify the rational function (see Chapter 6).

c Express $\frac{1}{x^3+x^2}$ in partial fractions.

First factorise;
$x^3 + x^2 = x^2(x+1)$. T
use $\frac{1}{x^2(x+1)} \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{}{x}$

Check that $\dfrac{1}{x^3+x^2} = \dfrac{1}{x^2} - \dfrac{1}{x} + \dfrac{1}{x+1}$

$$\int \frac{1}{x^3+x^2}\,dx = \int\left(\frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}\right)dx$$

$$= \int x^{-2}\,dx - \int \frac{1}{x}\,dx + \int \frac{1}{x+1}\,dx$$

The form $\int \frac{f'(x)}{f(x)}\,dx$.

$$= \frac{x^{-1}}{-1} - \ln|x| + \ln|x+1| + c$$

$$= \ln\left|\frac{x+1}{x}\right| - \frac{1}{x} + c$$

Example 3

a Find the values of a and b if $\displaystyle\int_{-1}^{1} \frac{1}{4-x^2}\,dx = \frac{1}{a}\ln b$.

b Evaluate $\displaystyle\int_0^4 \frac{26x^2+5x+31}{(1+3x)(x^2+16)}\,dx$.

Solution

a Notice that the denominator is the difference of two squares. That is, $4 - x^2 = (2+x)(2-x)$. Use this to write partial fractions.

Check that $\dfrac{1}{4-x^2} = \dfrac{1}{4(2+x)} + \dfrac{1}{4(2-x)}$

Use $\frac{1}{4-x^2} \equiv \frac{A}{2+x} + \frac{B}{2-x}$.

Now $\displaystyle\int_{-1}^{1} \frac{1}{4-x^2}\,dx = \int_{-1}^{1}\left[\frac{1}{4(2+x)} + \frac{1}{4(2-x)}\right]dx$

$$= \tfrac{1}{4}\int_{-1}^{1} \frac{1}{2+x}\,dx + \tfrac{1}{4}\int_{-1}^{1} \frac{1}{2-x}\,dx$$

Take the factor $\frac{1}{4}$ outs
the integral sign.

$$= \tfrac{1}{4}\int_{-1}^{1} \frac{1}{2+x}\,dx - \tfrac{1}{4}\int_{-1}^{1} \frac{-1}{2-x}\,dx$$

The form $\int \frac{f'(x)}{f(x)}\,dx$.

Use $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x$

$$= \tfrac{1}{4}[\ln|2+x|]_{-1}^{1} - \tfrac{1}{4}[\ln|2-x|]_{-1}^{1}$$

$$= \tfrac{1}{4}(\ln 3 - \ln 1) - \tfrac{1}{4}(\ln 1 - \ln 3)$$

$$= \tfrac{1}{4}\ln 3 - 0 - 0 + \tfrac{1}{4}\ln 3$$

$\ln 1 = 0$

$$= \tfrac{1}{2}\ln 3$$

Now $\quad \frac{1}{2}\ln 3 \equiv \frac{1}{a}\ln b$, so $a = 2$ and $b = 3$.

b Express the integrand in partial fractions.

Note the quadratic fa
in the denominator a
use
$\frac{26x^2+5x+31}{(1+3x)(x^2+16)} = \frac{A}{1+3x} + \frac{B}{x^2}$

Check that $\dfrac{26x^2+5x+31}{(1+3x)(x^2+16)} = \dfrac{2}{1+3x} + \dfrac{8x-1}{x^2+16}$

12.2 Rational Functions
Exercise
Technique

1 Integrate the following, first expressing each as a sum involving a proper fraction:

a $\dfrac{x}{x+1}$ **b** $\dfrac{x-1}{x+1}$ **c** $\dfrac{x}{2x-1}$ **d** $\dfrac{x^2}{x-1}$

2 Find the exact value of:

a $\displaystyle\int_0^{1/2} \frac{12x-1}{6x+1}\,dx$ **b** $\displaystyle\int_0^1 \frac{x^2-1}{x^2+1}\,dx$

3 Evaluate the following integrals, giving your answers correct to two decimal places:

a $\displaystyle\int_1^2 \frac{4x}{1+2x}\,dx$ **b** $\displaystyle\int_2^7 \frac{x}{3+x}\,dx$

4 Use partial fractions to find:

a $\displaystyle\int \frac{4x+7}{x^2+3x+2}\,dx$ **b** $\displaystyle\int \frac{5-11x}{x-2x^2}\,dx$

c $\displaystyle\int \frac{x+5}{x^2-1}\,dx$ **d** $\displaystyle\int \frac{4x-5}{(x+1)(x-2)}\,dx$

4 a

5 Find the exact value of:

a $\displaystyle\int_2^3 \frac{1}{x(x-1)}\,dx$ **d** $\displaystyle\int_3^4 \frac{9-3x}{(x-2)(5-x)}\,dx$

b $\displaystyle\int_2^6 \frac{6x}{(1+2x)(1-x)}\,dx$ **e** $\displaystyle\int_2^3 \frac{3}{(x-4)(x-1)}\,dx$

c $\displaystyle\int_2^4 \frac{x}{1-x^2}\,dx$ **f** $\displaystyle\int_3^4 \frac{5}{x(5-2x)}\,dx$

6 Find:

a $\displaystyle\int \frac{1-3x}{x(1+x^2)}\,dx$ **b** $\displaystyle\int \frac{3x^2+2x+4}{(x+1)(4+x^2)}\,dx$

7 Find the exact value of $\displaystyle\int_0^5 \frac{x^2-2x+24}{(1+2x)(25+x^2)}\,dx$.

8 Express the integrand as a sum involving a proper fraction before finding the exact value of $\displaystyle\int_0^1 \frac{2x^2+x-4}{x^2-x-2}\,dx$.

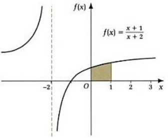
9 $f(x) = \dfrac{8}{(x+1)(x^2+3)}, x \neq 1$

a Express $f(x)$ in the form $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+3}$.

b Find the value of a such that $\displaystyle\int_0^1 f(x)\,dx = \ln a + \frac{\pi\sqrt{3}}{a^2}$.

Contextual

1

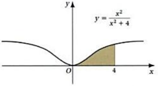


The graph shows the curve $f(x) = \dfrac{x+1}{x+2}$.

a Express $f(x)$ in the form $A + \dfrac{B}{x+2}$.

b Calculate the area of the shaded region.

2

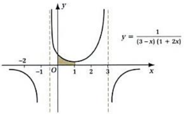


The graph shows the curve $y = \dfrac{x^2}{x^2+4}$.

a Find the values of A and B if $y \equiv A + \dfrac{B}{x^2+4}$.

b Calculate the area of the shaded region.

3



The graph shows the curve $y = \dfrac{1}{(3-x)(1+2x)}$. Find the shaded area.

Example 1

Find:

a $\int xe^{3x}\,dx$ **b** $\int x^2 \ln x\,dx$

Solution

a Let $u = x$ and $v' = e^{3x}$. Then $u' = 1$ and $v = \int e^{3x}\,dx = \frac{1}{3}e^{3x}$.

Now, $\int uv'\,dx = uv - \int vu'\,dx$

$$\int xe^{3x}\,dx = x \times \tfrac{1}{3}e^{3x} - \int\left(\tfrac{1}{3}e^{3x} \times 1\right)dx$$

$$= \tfrac{1}{3}xe^{3x} - \tfrac{1}{3}\int e^{3x}\,dx$$

$$= \tfrac{1}{3}xe^{3x} - \tfrac{1}{3} \times \tfrac{1}{3}e^{3x} + c$$ ◀ **Factorise.**

$$= \tfrac{1}{9}e^{3x}(3x - 1) + c$$

Choose which part o integrand is u and w is $\frac{dv}{dx}$ so that $\int v\frac{du}{dx}dx$ i simpler integral than original. See what w happen if we had ch $u = e^{3x}$ and $\frac{dv}{dx} = x$ in case.

Notice that the secon integral is much simp than the original.

b In this example the choice of u and v' are obvious because $\ln x$ cannot be integrated directly. However, it can be differentiated.
So let $u = \ln x$ and let $v' = x^2$. Then $u' = \frac{1}{x}$ and $v = \int x^2\,dx = \frac{1}{3}x^3$

Now, $\int uv'\,dx = uv - \int vu'\,dx$

So
$$\int x^2 \ln x\,dx = \ln \times \tfrac{1}{3}x^3 - \int\left(\tfrac{1}{3}x^3 \times \tfrac{1}{x}\right)dx$$

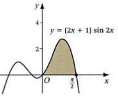
$$= \tfrac{1}{3}x^3 \ln x - \tfrac{1}{3}\int x^2\,dx$$

$$= \tfrac{1}{3}x^3 \ln x - \tfrac{1}{3} \times \tfrac{1}{3}x^3 + c$$

$$= \tfrac{1}{3}x^2 \ln x - \tfrac{1}{9}x^3 + c$$

$$= \tfrac{1}{9}x^3(3\ln x - 1) + c$$

Example 2



The graph shows part of the curve $y = (2x + 1)\sin 2x$. Find the area under the curve between $x = 0$ and $x = \frac{\pi}{2}$.

Sometimes integration by parts has to be used more than once in the same problem. Try finding $\int x^2 \sin x \, dx$. Integration by parts gives

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2\int x \cos x \, dx$$

Notice that the second integrand is a simpler product than the original one, but still cannot be integrated directly. However, it can be successfully integrated using integration by parts a second time giving:

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2\cos x + c$$

Check this result by differentiation.

The following example is an unusual demonstration of how integration by parts is used twice. When integration by parts is used for a second time the original integral appears again.

Example 4

Find $\int e^{2t} \sin t \, dt$.

Solution

$$\int u \frac{dv}{dt} dt = uv - \int v \frac{du}{dt} dt$$

Note the change of variable from x to t.

Let $u = e^{2t}$ and $\frac{dv}{dt} = \sin t$. Then $\frac{du}{dt} = 2e^{2t}$ and $v = \int \sin t \, dt = -\cos t$.

$$\text{Now} \quad \int e^{2t} \sin t \, dt = e^{2t} \times -\cos t - \int -\cos t \times 2e^{2t} \, dt$$

$$= -e^{2t} \cos t + 2\int e^{2t} \cos t \, dt$$

Using integration by parts again to integrate the second integral, let $u_2 = e^{2t}$ and $\frac{dv_2}{dt} = \cos t$. So $\frac{du_2}{dt} = 2e^{2t}$ and $v_2 = \int \cos t \, dt = \sin t$.

Use u_2 and v_2 so as n to confuse these functions with u and

$$\text{Now} \quad \int e^{2t} \sin t \, dt = -e^{2t} \cos t + 2\left(e^{2t} \sin t - \int \sin t \times 2e^{2t} \, dt\right)$$

$$\int e^{2t} \sin t \, dt = -e^{2t} \cos t + 2e^{2t} \sin t - 4\int e^{2t} \sin t \, dt$$

Notice that the integral now appears twice. The equation can be rearranged by treating this integral as a distinct term. Adding $4\int e^{2t} \sin t \, dt$ to both sides,

$$5\int e^{2t} \sin t \, dt = -e^{2t} \cos t + 2e^{2t} \sin t$$

$$\Rightarrow \quad \int e^{2t} \sin t \, dt = \tfrac{1}{5} e^{2t}(2\sin t - \cos t) + c$$

Remember to add a constant of integratio

Check this answer using the product rule. Differentiating $\frac{1}{5} e^{2t}(2\sin t - \cos t)$ should give the original integrand, $e^{2t} \sin t$.

12.4 Integration by Substitution

Integration by substitution is a technique used to transform difficult integrals into simpler ones in which the standard integrals can be used. How could $\int(1+3x)^6\,\mathrm{d}x$ be found quickly? We could expand $(1+3x)^6$ using the binomial expansion and integrate the resulting polynomial term by term. An alternative is to make the substitution $t = 1 + 3x$.

Use that technique to find $\int(1+3x)^6\mathrm{d}x$.

Then $\int(1+3x)^6\,\mathrm{d}x$ becomes $\int t^6\,\mathrm{d}x$.

Although the integral has been simplified we cannot integrate t^6 with respect to x. We need a $\mathrm{d}t$ rather than a $\mathrm{d}x$ after the integrand.

Let $\quad I = \int t^6\,\mathrm{d}x,\;$ then $\;\dfrac{\mathrm{d}I}{\mathrm{d}x} = t^6 \qquad$ [1]

Recall the fundamental theorem of calculus from Ch. 9.

The chain rule gives

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}I}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = t^6\frac{\mathrm{d}x}{\mathrm{d}t}$$

Integrating both sides with respect to t gives,

$$I = \int t^6\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$$

Comparing this to equation [1], we see that $\mathrm{d}x$ is effectively replaced by $\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$.

Since $t = 1 + 3x$ in this case, $\frac{\mathrm{d}t}{\mathrm{d}x} = 3$. So $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{3}$, and $\mathrm{d}x$ is replaced by $\frac{1}{3}\mathrm{d}t$.

$$\begin{aligned}
\text{So}\quad \int(1+3x)^6\,\mathrm{d}x &= \int t^6 \times \tfrac{1}{3}\mathrm{d}t \\
&= \tfrac{1}{3}\int t^6\mathrm{d}t \\
&= \tfrac{1}{3} \times \tfrac{1}{7}t^7 + c \\
&= \tfrac{1}{21}(1+3x)^7 + c
\end{aligned}$$

This can now be integrated.

Substitute $t = 1 + 3x$.

Notice that the substitution makes the integrand simpler. The difficult part is changing the variable with respect to which the integration is carried out from x to t.

Find the values of the new limits:

- when $x = 1$, $t = 1 - 1 = 0$
- when $x = 2$, $t = 1 - 2 = -1$

Do not worry that the upper limit is now less than the lower limit.

So $$\int_1^2 (1-x)^3\,dx = \int_0^{-1} t^3 \times (-1)\,dt$$
$$= -\int_0^{-1} t^3\,dt$$
$$= -[\tfrac{1}{4}t^4]_0^{-1}$$
$$= -[\tfrac{1}{4} - 0]$$
$$= -\tfrac{1}{4}$$

Replace dx with $\frac{dx}{dt}\,dt$, so here replace dx with $-dt$.

Integration by substitution can be used to establish other standard integrals. For example to find $\int (ax+b)^n\,dx$, where a, b and n are real numbers, use the substitution $t = ax + b$. Then $\frac{dt}{dx} = a$, and $\frac{dx}{dt} = \frac{1}{a}$.

Now $$\int (ax+b)^n\,dx = \int t^n \times \frac{1}{a}\,dt$$
$$= \frac{1}{a}\int t^n\,dt$$
$$= \frac{1}{a} \times \frac{t^{n+1}}{n+1} + c$$

Replace dx with $\frac{1}{a}$ dt.

Therefore

$$\boxed{\int (ax+b)^n\,dx = \frac{1}{a} \times \frac{(ax+b)^{n+1}}{n+1} + c}$$

◀ **Learn this result.**

Look back again at Example 1 where $(ax+b)^n = (5x+1)^{\frac{1}{2}}$. Here $a = 5$, $b = 1$ and $n = \frac{1}{2}$.

So $$\int (5x+1)^{\frac{1}{2}}\,dx = \frac{1}{5}\frac{(5x+1)^{\frac{3}{2}}}{(\frac{1}{2}+1)} + c$$
$$= \tfrac{2}{15}(5x+1)^{\frac{3}{2}} + c$$

Check the integrals in Examples 2 and 3 using this result.

More complex integrals require a higher level of algebraic skill to make sure that the integral has been expressed entirely in terms of the new variable only.

Example 4

Evaluate $\int_{-1}^{0} x\sqrt{1+x}\,dx$.

Solution

Let $t = 1 + x$. Then $\frac{dt}{dx} = 1$ and $\frac{dx}{dt} = 1$.

When $x = -1$, $t = 0$ and when $x = 0$, $t = 1$.

Now $\displaystyle\int_{-1}^{0} x\sqrt{1+x}\,dx = \int_{0}^{1} (t-1)\sqrt{t} \times 1\,dt$

$$= \int_{0}^{1} (t^{3/2} - t^{1/2})\,dt$$
$$= [\tfrac{2}{5}t^{5/2} - \tfrac{2}{3}t^{3/2}]_{0}^{1}$$
$$= (\tfrac{2}{5} \times 1 - \tfrac{2}{3} \times 1) - (0 - 0)$$
$$= \tfrac{2}{5} - \tfrac{2}{3}$$
$$= -\tfrac{4}{15}$$

Replace dx with dt.

Notice that the integran $x\sqrt{1+x}$ is written in terms of t only; $x\sqrt{1+x} = (t-1)\sqrt{t}$.

Example 5

Use the substitution $u^2 = 5 - x$ to find $\int_1^5 (x+1)(5-x)^{\frac{1}{2}}\,dx$.

Solution

Let $u^2 = 5 - x$, and differentiate implicitly with respect to x.
Then $2u\frac{du}{dx} = -1$ and so $\frac{dx}{du} = -2u$.

When $x = 1$, $u^2 = 5 - 1 = 4 \Rightarrow u = 2$
When $x = 5$, $u^2 = 5 - 5 = 0 \Rightarrow u = 0$

Also, $x = 5 - u^2 \Rightarrow x + 1 = 6 - u^2$

Now $\displaystyle\int_{1}^{5} (x+1)(5-x)^{\frac{1}{2}}\,dx = \int_{2}^{0} (6-u^2)(u^2)^{\frac{1}{2}} \times -2u\,du$

$$= -2\int_{2}^{0} (6-u^2)u^2\,du$$
$$= -2\int_{2}^{0} (6u^2 - u^4)\,du$$
$$= -2[2u^3 - \tfrac{1}{5}u^5]_{2}^{0}$$
$$= -2[(0-0) - (2 \times 2^3 - \tfrac{1}{5} \times 2^5)]$$
$$= -2(-16 + \tfrac{32}{5})$$
$$= -2(-\tfrac{48}{5})$$
$$= \tfrac{96}{5}$$
$$= 19\tfrac{1}{5}$$

Replace dx with $-2u$ du

An alternative method i to use the result $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. In this case we would evaluate $2\int_0^2 (6u^2 - u^4)$ o

Choosing suitable substitutions

Try finding $\int xe^{x^2}\,dx$. Using a systematic approach, integration by parts may be your first choice, but the second integral is too difficult to integrate. A good strategy is then to try a substitution.

Spotting the substitution $t = g(x)$, where $g(x)$ is some function of x, often depends on recognising a common factor that appears in the integrand and in $g'(x)$.

Step ① Let t be some function of x so that a factor is created that will cancel or simplify the integrand.

Let $t = x^2$ ◀ **Differentiating $t = x^2$ wrt x gives a factor of x, which appears in the integrand.**

Step ② Write dx in terms of dt.

$$\frac{dt}{dx} = 2x \Rightarrow dx = \frac{1}{2x}dt$$

Now $\int xe^{x^2}\,dx = \int xe^t \times \frac{1}{2x}dt$ ◀ **Notice that the x will cancel.**

$$= \tfrac{1}{2}\int e^t\,dt$$

At this stage some algebra may be necessary to write the integrand in terms of the new variable t.

Step ③ Integrate with respect to t.

$$\int xe^{x^2}\,dx = \tfrac{1}{2}\int e^t\,dt$$
$$= \tfrac{1}{2}e^t + c$$

Step ④ Substitute for t throughout to express the integral in terms of the original variable, x.

$$\int xe^{x^2}\,dx = \tfrac{1}{2}e^{x^2} + c$$

Check this result using differentiation.

Other useful hints

- For integrals containing e^x, try $t = e^x$.
- For integrals containing $\sqrt{a^2 - x^2}$, try $x = a\sin\theta$. For example, with $\sqrt{4 - x^2}$, use $x = 2\sin\theta$.
- For integrals containing $\sqrt{a^2 + x^2}$, try $x = a\tan\theta$. For example, with $\sqrt{9 + x^2}$, use $x = 3\tan\theta$.

Example 6

Find:

a $\int \sin^2\theta\cos\theta\,d\theta$ **b** $\int \frac{e^x + e^{2x}}{1 + e^{2x}}\,dx$

Solution

a Let $t = \sin\theta$. ◀ **① Differentiating $t = \sin\theta$ gives us a factor of $\cos\theta$.**

Then $\frac{dt}{d\theta} = \cos\theta \Rightarrow d\theta = \frac{1}{\cos\theta}dt$. ◀ **② Write $d\theta$ in terms of dt.**

Now $\int \sin^2\theta \cos\theta \, d\theta = \int t^2 \times \cos\theta \times \frac{1}{\cos\theta} dt$

$= \int t^2 \, dt$

$= \frac{1}{3}t^3 + c$ ◀ ③ **Integrate wrt t.**

$= \frac{1}{3}\sin^3\theta + c$ ◀ ④ **Substitute for t.**

Notice that the $\cos\theta$ terms cancel.

b Let $p = e^x$. ◀ ① **Exponential functions in the integrand can be simplified.**

Then $\frac{dp}{dx} = e^x \Rightarrow dx = \frac{1}{e^x} dp$ ◀ ② **Write dx in terms of dp.**

e^x and e^{2x} can then be replaced by p and p^2.

Now, $\int \frac{e^x + e^{2x}}{1 + e^{2x}} dx = \int \frac{p + p^2}{1 + p^2} \times \frac{1}{e^x} dp$

$= \int \frac{p + p^2}{1 + p^2} \times \frac{1}{p} dp$

$= \int \frac{1 + p}{1 + p^2} dp$

$= \int \frac{1}{1 + p^2} dp + \int \frac{p}{1 + p^2} dp$ ◀ ③ **Integrate wrt p.**

$= \int \frac{1}{1 + p^2} dp + \frac{1}{2}\int \frac{2p}{1 + p^2} dp$

$= \tan^{-1}(p) + \frac{1}{2}\ln(1 + p^2) + c$ ◀ ④ **Substitute for p.**

$= \tan^{-1}(e^x) + \frac{1}{2}\ln(1 + e^{2x}) + c$

The $\frac{1}{p}$ term cancels with the common factor of p in the numerator.

Use $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$. Remember that $1 + p^2 > 0$ for all p

Example 7

Evaluate:

a $\int_0^2 \frac{x}{\sqrt{2x^2 + 1}} dx$ **b** $\int_0^{\frac{3}{2}} \sqrt{3 - x^2} \, dx$

Solution

a Let $u = 2x^2 + 1$. ◀ ① **Differentiating $u = 2x^2 + 1$ gives us a factor of x.**

Then $\frac{du}{dx} = 4x \Rightarrow dx = \frac{1}{4x} du$ ◀ ② **Write dx in terms of du.**

Find the limits in terms of u.

When $x = 0$, $u = 1$ and when $x = 2$, $u = 9$

Now, $\int_0^2 \frac{x}{\sqrt{2x^2 + 1}} dx = \int_1^9 \frac{x}{\sqrt{u}} \times \frac{1}{4x} du$

$= \frac{1}{4}\int_1^9 u^{-1/2} \, du$ ◀ ③ **Integrate wrt u.**

$= \frac{1}{4}[2u^{1/2}]_1^9$

$= \frac{1}{4}(2\sqrt{9} - 2\sqrt{1})$

$= \frac{1}{2}(3 - 1)$

$= 1$

Notice that the x terms cancel.

12.5 The Area Under a Parametrically Defined Curve

In Chapter 9 we developed a technique for finding the area under a curve. The equation of the curve was expressed in Cartesian form as $y = f(x)$ and the area calculated as the integral $\int_a^b f(x)\,dx$. The limits $x = a$ and $x = b$ define the boundaries of the area along the x-axis.
What happens when the curve is expressed in parametric form? Instead of $y = f(x)$ we have $y = g(t)$ for some parameter t. Since the integrand is now a function of the parameter t, we cannot use the operator dx. Instead we have to make a substitution and replace dx by $\frac{dx}{dt}\,dt$. Also, since the integrand and operator are expressed in terms of the parameter, so are the limits.

Example 1

A circle can be described by the parametric equations

$$x = 3\cos\theta \qquad y = 3\sin\theta \qquad 0 \leq \theta \leq 2\pi$$

a Find the area of the first quadrant.
b Verify the answer to **a** using the result

$$\text{area} = \int_{t_1}^{t_2} y(t)\,\frac{d}{dt}(x)\,dt$$

Solution

a $x = 3\cos\theta$, $y = 3\sin\theta$, $0 \leq \theta \leq 2\pi$ are the parametric equations of a circle, centre the origin and radius 3 (see p. 473).

$$\begin{aligned}\text{area of quadrant} &= \tfrac{1}{4} \times \pi r^2\\ &= \tfrac{1}{4} \times \pi \times 3^2\\ &= \frac{9\pi}{4}\end{aligned}$$

b To verify this result, notice that the first quadrant is described by parametric equations for $0 \leq \theta \leq \frac{\pi}{2}$.

$$\text{area} = \int_{t_1}^{t_2} y(t)\,\frac{d}{dt}(x)\,dt$$

To find the values of t_1 and t_2, the lower and upper limits on the integral in terms of the parameter t, we first need to consider the corresponding values of x. The first quadrant of the circle is bounded by $x = 0$ and $x = 3$. When $x = 0$, $t = \frac{\pi}{2}$ and when $x = 3$, $t = 0$. This means that $t_1 = \frac{\pi}{2}$ and $t_2 = 0$.

$$\text{Area} = \int_{\frac{\pi}{2}}^{0} 3\sin\theta \frac{\mathrm{d}}{\mathrm{d}\theta}(3\cos\theta)\,\mathrm{d}\theta$$

$$= \int_{\frac{\pi}{2}}^{0} 3\sin\theta(-3\sin\theta)\,\mathrm{d}\theta$$

$$= -9\int_{\frac{\pi}{2}}^{0} \sin^2\theta\,\mathrm{d}\theta$$

To integrate $\sin^2\theta$, recall that we can use the double angle formula from Chapter 3. This gives

$$\int \sin^2 x\,\mathrm{d}x = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + c$$

See p. 391.

So

$$\text{area} = -9\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_{\frac{\pi}{2}}^{0}$$

$$= -9\left[0 - \left(\frac{\pi}{4} - \tfrac{1}{4}\sin\pi\right)\right]$$

$$= \frac{9\pi}{4}$$

Recall special angles: $\sin\pi = 0$.

Example 2

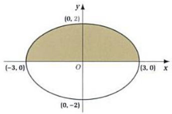
An ellipse can be described by the parametric equations

$$x = 3\cos\theta \qquad y = 2\sin\theta \qquad 0 \le \theta \le 2\pi$$

Calculate the area of the ellipse that is above the x-axis.

Solution

The parametric equations $x = 3\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le 2\pi$, define an ellipse centred on the origin. The ellipse crosses the x-axis at $x = -3$ and $x = +3$, and crosses the y-axis at $y = -2$ and $y = 2$.



$$\text{area} = \int_{t_1}^{t_2} y(t)\frac{\mathrm{d}}{\mathrm{d}t}(x)\,\mathrm{d}t$$

In parametric form this integral becomes

$$V = \int_{t_1}^{t_2} \pi (y(t))^2 \frac{\mathrm{d}}{\mathrm{d}t}(x)\,\mathrm{d}t$$

So

$$\begin{aligned}
\text{Volume} &= \int_1^4 \pi \left(\frac{4}{t}\right)^2 \frac{\mathrm{d}}{\mathrm{d}t}(4t)\,\mathrm{d}t \\
&= \int_1^4 \pi \frac{16}{t^2} \times 4\,\mathrm{d}t \\
&= 64\pi \int_1^4 \frac{1}{t^2}\,\mathrm{d}t \\
&= 64\pi \left[-\frac{1}{t}\right]_1^4 \\
&= 64\pi(-\tfrac{1}{4} + 1) \\
&= 48\pi
\end{aligned}$$

Now consider the first order differential equation $\frac{dy}{dx} = xy$. The algebraic solution cannot be found by simply integrating both sides with respect to x. Instead we must first separate the variables so that all the y terms appear on the same side of the equation as $\frac{dy}{dx}$, and all the x terms appear on the other side.

Step ① Separate the variables.

Since $\frac{dy}{dx} = xy,\quad \frac{1}{y}\frac{dy}{dx} = x$

Step ② Integrate both sides with respect to x.

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int x\,dx$$

$$\int \frac{1}{y}\,dy = \int x\,dx$$

Recall that $\frac{dy}{dx}\,dx$ can be replaced with dy.

Notice that the terms have been separated so that all the x terms will be integrated with respect to x and all the y terms with respect to y.

Thus $\ln|y| = \frac{1}{2}x^2 + c$, giving $|y| = e^{\frac{1}{2}x^2 + c}$

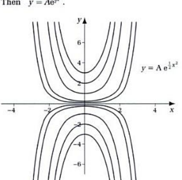
Therefore $y = \pm e^{\frac{1}{2}x^2 + c}$ or $y = \pm e^{\frac{1}{2}x^2} \times e^c$

Use law of indices $a^{m+n} = a^m \times a^n$.

Step ③ Simplify the constant (or find it if enough information is given in the question).

Since if c is a constant then e^c is also a constant, we can write $A = \pm e^c$.

Then $y = Ae^{\frac{1}{2}x^2}$.



If $A > 0$ the curves above the x-axis are generated. If $A < 0$ the curves below the x-axis are generated.

A particular solution can be found if more information is given. This is called the **variable separable** method of solving first order differential equations.

Example 1

Find the general solutions to:

a $\frac{dy}{dx} = xy^2$ **b** $\frac{dy}{dx} - 2y = 1$

Solution

a $\frac{1}{y^2}\frac{dy}{dx} = x$ ◀ ① **Separate the variables.**

$\int \frac{1}{y^2}\frac{dy}{dx}\,dx = \int x\,dx$ ◀ ② **Integrate both sides wrt x.**

$\frac{dy}{dx}\,dx$ is replaced wit[h] dy.

$\int y^{-2}dy = \int x\,dx$

$\frac{y^{-1}}{-1} = \frac{1}{2}x^2 + c$

Only one constant of integration is needed.

$-\frac{1}{y} = \frac{1}{2}(x^2 + 2c) = \frac{x^2 + 2c}{2}$

If c is a constant then is $2c$; let $k = 2c$.

$y = -\frac{2}{x^2 + k}$ ◀ ③ **Simplify the constant.**

This is the general solution.

The general solution be in the form '$y =$ '.

b $\frac{dy}{dx} = 2y + 1$

$\Rightarrow \quad \frac{1}{2y+1}\frac{dy}{dx} = 1$ ◀ ① **Separate the variables.**

$\frac{dy}{dx}\,dx$ is replaced wit[h] dy.

$\int \frac{1}{2y+1}\frac{dy}{dx}\,dx = \int dx$ ◀ ② **Integrate both sides wrt x.**

$\frac{1}{2}\int \frac{2}{2y+1}dy = \int dx$ ◀ **The form** $\int \frac{f'(x)}{f(x)}\,dx$.

$\frac{1}{2}\ln|2y+1| = x + c$

$\ln|2y+1| = 2x + 2c$

Let $k = 2c$.

$\Rightarrow \quad |2y+1| = e^{2x+k}$

$2y + 1 = \pm e^{2x} \times e^k$

$\Rightarrow \quad 2y = Ae^{2x} - 1$

Let $A = \pm e^k$.

$y = \frac{1}{2}Ae^{2x} - \frac{1}{2}$ ◀ ③ **Simplify the constant.**

So the general solution is $y = Be^{2x} - \frac{1}{2}$, where B is a constant.

Example 2

Find the solution of the differential equation $(1 + x)\frac{dy}{dx} = 1 - \sin^2 y$, for which $y = \frac{\pi}{4}$ when $x = 0$.

(OCSEB)

Solution

$$\frac{1}{1-\sin^2 y}\frac{dy}{dx} = \frac{1}{1+x}$$ ◀ ① **Separate the variables.**

$$\int \frac{1}{1-\sin^2 y}\frac{dy}{dx}\,dx = \int \frac{1}{1+x}\,dx$$ ◀ ② **Integrate both sides wrt x.**

$$\int \frac{1}{\cos^2 y}\,dy = \int \frac{1}{1+x}\,dx$$

$$\int \sec^2 y\,dy = \int \frac{1}{1+x}\,dx$$

$$\tan y = \ln|1+x| + c$$

Use the Pythagorean identity. $\sin^2 y + \cos^2 y = 1$.

When $x = 0$, $y = \frac{\pi}{4}$.

Thus $\tan\frac{\pi}{4} = \ln 1 + c$ ◀ ③ **Find the constant.**

$$1 = 0 + c$$

$$\Rightarrow 1 = c$$

So $\tan y = \ln|1+x| + 1$

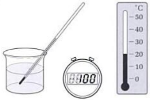
$$y = \tan^{-1}(\ln|1+x| + 1)$$

Modelling with differential equations

When there is sufficient information to describe how one variable changes with respect to another, the situation can often be modelled by a differential equation.

Newton's law of cooling

This states that the rate of change in temperature of an object is proportional to the *difference* between its temperature and that of its surroundings. Newton's law can be modelled by a differential equation.



For hot water cooling, let θ be the temperature of the water at time t minutes, and the surrounding temperature be 18°C. The difference between them is then $\theta - 18$. Then Newton's law gives

$$\frac{d\theta}{dt} \propto (\theta - 18)$$

To model a real situation some assumptions have to be made. In this case it is assumed that the temperature of the surroundings is constant.

Writing this as an equation,

$$\frac{d\theta}{dt} = -k(\theta - 18), \quad \text{where } k \in \mathbb{R}.$$

The negative sign indicates that θ is decreasing. If it were to be omitted the constant k would work out to be negative.

Population growth

Scientists have crudely suggested that the populations of certain organisms grow at a rate that is proportional to the size of the population. Let N be the population at time t. Then

$$\frac{dN}{dt} \propto N$$

Writing this as an equation,

$$\frac{dN}{dt} = kN, \quad \text{where} \quad k \in \mathbb{R}.$$

Here it is assumed th the environment rem constant. The model probably more realis over a short period o time.

Example 3

According to Newton's Law, the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. Warm water is poured into a basin, and cools from 41.1°C to 40.0°C in 5 minutes. The temperature of the room is a constant 17°C.

a Write down a differential equation to model the temperature of the water.

b Solve the differential equation, and find the temperature of the water after 10 minutes.

c Find the time taken for the water to cool to 37°C.

Graphi calcula suppor pack

Solution

a Let θ be the temperature in degrees Celsius of the water at time t. Then according to Newton's Law the rate of change of θ with respect to time is proportional to $\theta - 17$.

$$\frac{d\theta}{dt} \propto (\theta - 17)$$

$$\frac{d\theta}{dt} = -k(\theta - 17), \text{ where } k \text{ is a constant}$$

Remember that the negative sign indicat cooling.

b $\dfrac{1}{\theta - 17}\dfrac{d\theta}{dt} = -k$ ◄ ① **Separate the variables.**

$$\int \frac{1}{\theta - 17}\, d\theta = \int -k\, dt$$ ◄ ② **Integrate both sides wrt t.**

Since $\theta > 17$ the modulus function isı required.

$$\ln(\theta - 17) = -kt + c$$

$$\theta - 17 = e^{-kt+c}$$

$$\theta - 17 = e^{-kt} \times e^{c}$$

$$\theta - 17 = Ae^{-kt}, \text{ where } A \text{ is a constant}$$

$$\theta = 17 + Ae^{-kt}$$

12.6 Differential Equations Exercise

Technique

1 Find the general solution to the following differential equations:

a	$\frac{dy}{dx} = 3x^2 + \frac{1}{x}$	**d**	$\frac{dy}{dx} = \sqrt{x} + 2e^x$	**g**	$2 + x = x^2\frac{dy}{dx}$
b	$\frac{dy}{dx} = x - \frac{1}{x} + e^x$	**e**	$\frac{dy}{dx} = x^{\frac{1}{3}} + \sin x$	**h**	$\frac{1}{2}e^x + 3\frac{dy}{dx} = 1$
c	$\frac{dy}{dx} = x + e^x$	**f**	$x\frac{dy}{dx} = 1 - xe^x$	**i**	$10\frac{dy}{dx} = x^{\frac{3}{4}} + \frac{20}{x}$

2 Find the general solution to the following differential equations.

a	$\frac{dy}{dx} = 12x^2$	**e**	$\frac{dy}{dx} = 2y$	**i**	$\frac{dy}{dx} - y = 1$
b	$\sec x\frac{dy}{dx} = 1$	**f**	$\frac{dy}{dx} = 6x^2y$	**j**	$(x+1)\frac{dy}{dx} - 2y = 0$
c	$\frac{dy}{dx} = \frac{x+1}{y}$	**g**	$\frac{dy}{dx} = y^{\frac{2}{3}}$	**k**	$\frac{dy}{dx} = \frac{x^2}{y}$
d	$\frac{dy}{dx} = \frac{1}{y}$	**h**	$\frac{dy}{dx} = xe^{-y}$	**l**	$\frac{dy}{dx} = \frac{xy}{1+x^2}$

2f

3
- **a** Find $\int xe^x\,dx$.
- **b** Use your result to **a** to find the general solution to $e^{-x}\frac{dy}{dx} = \frac{x}{y}$.

4 The variables x and y satisfy the differential equation $\frac{dy}{dx} + y\tan x = 0$.

- **a** Find the general solution.
- **b** Find a particular solution where $y = 1$ when $x = \frac{\pi}{3}$.

5
- **a** Find the general solution to $\frac{dy}{dx} = \frac{2(y-1)}{x}$.
- **b** Find a particular solution where $y = 4$ when $x = 1$.

5

Contextual

1 An object cools from 100°C to 70°C in 34 minutes. Assume Newton's law of cooling applies. The surrounding air temperature is 20°C.

- **a** State Newton's Law of cooling.
- **b** If T is the temperature of the body at time t minutes, write down a differential equation that describes how the body cools.
- **c** Solve the differential equation to find T in terms of t and use it to find the temperature after 45 minutes.
- **d** How long will the body take to cool to 50°C?

2 The rate of increase of the number of bacteria present in a culture is proportional to the number present at that time.

- **a** If N represents the number of bacteria after t hours write down a differential equation connecting N and t.

Consolidation

Exercise A

1 Integrate the following with respect to x:

a	$\frac{1}{x^2}$	**e**	$\sin 5x \sin x$	**i**	$\cot 3x$
b	$4\cos(2x+7)$	**f**	$1 + e^{3x}$	**j**	$\frac{1}{25+x^2}$
c	$\sin 2x + \cos \frac{1}{2}x$	**g**	$\frac{1}{e^{5x}}$	**k**	$\frac{1}{\sqrt{25-x^2}}$
d	$\sec^2 4x$	**h**	$\frac{4x+6}{x^2+3x}$	**l**	$\frac{x}{x+3}$

2 Use integration by parts to find the indefinite integral $\int x^4 \ln x \, dx$. *(SMP)*

3 Evaluate $\int_0^1 xe^{3x} \, dx$. *(ULEAC)*

4 Show that $\displaystyle\int_0^1 \frac{1}{(x+1)(2x+1)} \, dx = \ln a$, where a is a rational number to be determined. *(JMB)*

5 Find the values of a and b if $\displaystyle\int_0^2 \frac{3}{x^2-9} \, dx = -\frac{1}{a} \ln b$.

6 Use the substitution $u = 1 + 3\sin\theta$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{1+3\sin\theta}} \, d\theta$. *(SMP)*

7 Find the value of $\int_3^4 x(x-3)^{17} \, dx$. *(ULEAC)*

8 Find y in terms of x given that $\frac{dy}{dx} = ye^x$ and that $y = 1$ when $x = 0$. *(ULEAC)*

9 A balloon is expanding and at time t seconds its surface area is s cm. The expansion is such that the rate of increase of s is proportional to $\frac{1}{\sqrt{s}}$. When the surface area is $900\,\text{cm}^2$, it is increasing at a rate of $60\,\text{cm}^2\,\text{s}^{-1}$.

a Show that $\frac{ds}{dt} = \frac{1800}{\sqrt{s}}$.

b Solve this differential equation given that $s = 400$ when $t = 0$.

c Find to the nearest second the time at which $s = 900$ cm.

(NEAB)

10 At time t hours the rate of decay of the mass of a radioactive substance is proportional to the mass x kg of the substance at that time. At time $t = 0$ the mass of the substance is A kg.

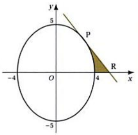
a By forming and integrating a differential equation, show that $x = Ae^{-kt}$, where k is a constant.

b It is observed that $x = \frac{1}{3}A$ at time $t = 10$. Find the value of t when $x = \frac{1}{2}A$, giving your answer to two decimal places. *(ULEAC)*

Exercise B

1 The curve shown has parametric equations $x = 4\cos\theta$, $y = 5\sin\theta$, $0 \leq \theta \leq 2\pi$.

The straight line PR is a tangent to the ellipse at P, where $\theta = \frac{\pi}{3}$. Find the area of the shaded region, bound by the curve, the x-axis and the tangent PR.



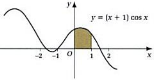
2 Evaluate the following integrals exactly:

a $\displaystyle\int_0^{\frac{\pi}{2}} \sin x \cos 3x \, dx$

b $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} \, dx$

3 a Use integration by parts to find $\int xe^{-x} \, dx$.

b



The graph shows part of the curve $y = (x+1)\cos x$. Calculate, correct to two decimal places, the area of the shaded region.

4 a Express $\dfrac{4x^2+2x+1}{(x+1)(2x-1)}$ in the form $A + \dfrac{B}{(x+1)(2x-1)}$. Use this to find $\displaystyle\int \frac{4x^2+2x+1}{(x+1)(2x-1)} \, dx$.

b Express $\dfrac{3x^2+3x+4}{x(x^2+4)}$ in the form $\dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$. Then find $\displaystyle\int \frac{3x^2+3x+4}{x(x^2+4)} \, dx$.

c Evaluate $\displaystyle\int_0^2 \frac{x(13-3x)}{(x+1)(x-3)^2} \, dx$.

5 a By means of the substitution $u = 6 - x$ or otherwise, show that $\int \frac{1}{6-x} \, dx = -\ln|6-x| + c$.

b As a mathematical model for my regular 6 km morning jog, I assume that my speed is proportional to the distance that I still have to go to reach the end.

i If I start off at a speed of $8\,\text{km}\,\text{h}^{-1}$, and after t hours have travelled x km, explain why $\frac{dx}{dt} = k(6 - x)$, where k is a constant.

ii Show that $k = \frac{4}{3}$.

c By solving the differential equation $\frac{dt}{dx} = \frac{3}{4(6-x)}$, find the time taken for me to complete 4 km of the jog, and state my predicted speed at the time.

d By considering the predicted time taken for me to complete the jog, comment on the suitability of the model.

(SMP)

6 Water flows out of a tank through a pipe at the bottom, and at a time t minutes the depth of water in the tank is x metres. At time $t = 0$ the depth of water in the tank is 2 m. After 10 minutes the depth is 1.5 m. It is desired to find the time T, correct to the nearest minute, at which the depth is 1 m.

a In a simple model the rate at which the depth of water in the tank decreases is taken to be constant. Find T using this model.

b In a more refined model the rate at which the depth of water remaining in the tank is decreasing at any instant is proportional to the square root of the depth at that instant.

i Explain how the more refined model leads to the differential equation $-\frac{dx}{dt} = k\sqrt{x}$, where k is a positive constant.

ii Find the general solution of the differential equation in **i**.

iii Find T using the more refined model.

(UCLES)

7 A mathematical model for the population growth of a certain country assumes that the country has a constant birth rate of 22 per thousand and a constant death rate of 7 per thousand. The population of the country at the end of 1995 was 4.80 million and the population of the country at time t years later is denoted by x.

a Show that $\frac{dx}{dt} = 0.015x$.

b Find x in terms of t.

Use the mathematical model to estimate:

c the population of the country at the end of 1998

d the year in which the population of the country will exceed 6 million.

(WJEC)

Applications and Activities

1 *Cooling*

Fill a container with hot water. Note the room temperature and place a thermometer in the hot water and wait until it has adapted to the temperature of the water. At this point begin to note the water temperature every $2\frac{1}{2}$ minutes. Record your results in a table similar to the one below.

Time (t)	0	2.5	5	7.5	10	12.5	15	17.5	20	22.5	25	27.5
Temp. (θ°C)												

a Write down Newton's law of cooling

b Write down a differential equation that will model the temperature of the water.

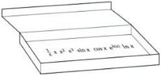
c Solve the differential equation and use the first and third entries in the table to find θ in terms of t.

d Investigate how well your differential equation models this situation.

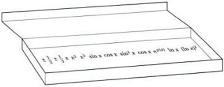
e Investigate Newton's law of cooling for different shaped containers.

2 *Integration spotting*

The fundamental theorem of calculus states that differentiation is the reverse of integration. In this activity an integral in the form $\int f(x)g(x)\,\mathrm{d}x$ must be made by selecting two function from the box below.



The answer must contain a combination of terms from the second box, below.



One example is $\int \frac{1}{x} \times x^2\,\mathrm{d}x = \frac{1}{2}x^2 + c$.
A more complex example is $\int xe^{x^2}\,\mathrm{d}x = \frac{1}{2}e^{x^2} + c$, because $\frac{\mathrm{d}}{\mathrm{d}x}(e^{x^2} + c) = 2xe^{x^2}$.
How many different integrals complete with answers can you spot?

Summary

- Standard integrals that are the reverse of differentiating standard functions are:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \; n \neq -1$$

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + c$$

$$\int \cos kx \, dx = \frac{1}{k} \sin kx + c$$

$$\int \frac{1}{x} \, dx = \ln x + c$$

$$\int \sin kx \, dx = -\frac{1}{k} \cos kx + c$$

$$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sec^2 kx \, dx = \frac{1}{k} \tan kx + c$$

$$\int \frac{a}{a^2 + x^2} \, dx = \tan^{-1}\left(\frac{x}{a}\right) + c$$

- If the integrand has $\cos^2 x$ or $\sin^2 x$ terms, use the double angle formulas before integrating.
- If the integrand is a product of trigonometrical functions, use the compound angle formulas to transform the integrand into a sum of trigonometrical functions.
- Integrate rational functions by first expressing them in proper fraction form and then use partial fractions if required.
- Use integration by parts to integrate a product of functions:

$$\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$$

- Substitute suitable functions to simplify the integrand.
- The area under a parametrically defined curve is given by $\int_{t_1}^{t_2} y(t) \frac{d}{dt}(x) \, dt$, where t_1 and t_2 are the values of the parameter t corresponding to the lower and upper values of x.
- Solve a differential equation by separating the variables to find either a general solution or a particular solution.
- Newton's law of cooling and rate of change of population are both modelled by differential equations.

13.1 Approximate Solutions of Equations

Try solving the equations $2\cos x - x + 1$ and $e^x = x + 3$. If you tried to solve them using algebraic techniques the temptation is to give up and say the equations cannot be solved. However, graphical, or **iterative** methods can be used to find an approximate value for the solution to these equations. These techniques generate numerical values of increasing accuracy. They are based on repeating a procedure or **algorithm** until sufficient accuracy has been achieved.

Graphical methods

The root(s) of any equation of the form $f(x) = 0$ can be found by drawing the graph of $y = f(x)$ and reading off the approximate values of x for which $y = 0$.

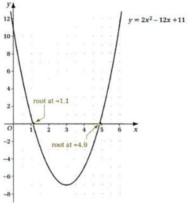
$f(x) = 0$ where the gr
crosses the x-axis.

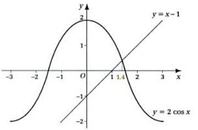
Example 1

Construct a table of values for $y = 2x^2 - 12x + 11$ with $0 \le x \le 6$. Draw the graph of $y = 2x^2 - 12x + 11$ in that range, and use it to find approximate solutions of the equation $2x^2 - 12x + 11 = 0$, to one decimal place.

Solution

x	0	1	2	3	4	5	6
y	11	1	−5	−7	−5	1	11





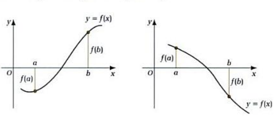
The graph of $y = 2\cos$ [cut off] is periodic, oscillatin[g] between maximum a[nd] minimum values of 2 and −2 respectively. [The] graph of $y = x - 1$ is [a] straight line.

There is only one point of intersection. Since the graphs of $y = 2\cos x$ and $y = x - 1$ do not intersect anywhere else, the equation $2\cos x - x + 1 = 0$ has only one solution. The x-coordinate at the point of intersection is about 1.4, so the approximate value of the root to this equation is $x = 1.4$ (1 d.p.).

Use a graphical calculator to plot the graphs of $y = 2\cos x$ and $y = x - 1$ on the same axes, and use the TRACE function to confirm that $x = 1.4$ is the approximate solution of the equation $2\cos x = x - 1$. Remember to work in radians.

Investigate whether y[our] calculator has an INTERSECT function that will find values [of x] at which the two grap[hs] cross.

Locating roots by a 'change of sign'



Suppose $f(x)$ is a continuous function, defined for all points between $x = a$ and $x = b$. If $f(a)$ and $f(b)$, the values of the function at $x = a$ and $x = b$, respectively, are opposite in sign to each other, then $f(a) < 0$ and $f(b) > 0$, or vice versa. This is illustrated in the diagrams above. Since the graph changes from negative to positive, or from positive to negative, somewhere between a and b, there must be a solution to the equation $f(x) = 0$ in the interval $a < x < b$.

So by looking for a change of sign in the value of $f(x)$ between nearby values of x, the approximate location of the roots of the equation $f(x) = 0$ can be found.

This means that

$$\frac{x_1 - a}{p} = \frac{b - x_1}{q}$$

where p and q are the absolute values of $f(a)$ and $f(b)$, and x_1 is the approximation to the root α. This equation can now be rearranged to make x_1 the subject.

$$(x_1 - a)q = (b - x_1)p$$

$$x_1 q - aq = bp - x_1 p$$

$$x_1 p + x_1 q = aq + bp$$

$$x_1(p + q) = aq + bp$$

$$\boxed{x_1 = \frac{aq + bp}{p + q}}$$

◀ **Learn this important result.**

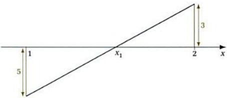
This formula can now be used to find the first and subsequent approximation to the root of the equation.

Example 5

The equation $x^3 + x - 7 = 0$ has a root in the interval $1 < x < 2$. Use linear interpolation to find the approximate value of this root, correct to two decimal places.

Solution

Let $f(x) = x^3 + x - 7$. Check that $f(1) = -5$ and $f(2) = 3$.



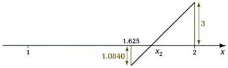
The curve has not be drawn here. We only need to know the absolute values of $f($ and $f(2)$.

Now use the formula $x_1 = \frac{aq+bp}{p+q}$ with $a = 1$, $b = 2$, $p = 5$ and $q = 3$. This will calculate the value of x_1 at which the interval $1 < x < 2$ is divided in linear proportion,

$$x_1 = \frac{(1 \times 3) + (2 \times 5)}{5 + 3}$$
$$= \tfrac{13}{8}$$
$$= 1.625$$

Now $f(1.625) = -1.0840$ (4 d.p.)

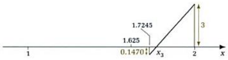
Notice that there is a change of sign in the value of $f(x)$ between $x = 1.625$ and $x = 2$. This means that the root lies in the interval $1.625 < x < 2$, so now repeat the process using the same formula, with $a = 1.625$, $b = 2$, $p = 1.0840$ and $q = 3$.



$$x_2 = \frac{(1.625 \times 3) + (2 \times 1.0840)}{1.0840 + 3}$$
$$= 1.7245 \text{ (4 d.p.)}$$

Now $f(1.7245) = -0.1470$ (4 d.p.)

There is a change of sign in the value of the function $f(x)$ between $x = 1.7245$ and $x = 2$ so the root lies in the interval $1.7245 < x < 2$. Repeat the process again, using the formula, with $a = 1.7245$, $b = 2$, $p = 0.1470$ and $q = 3$.



$$x_3 = \frac{(1.7245 \times 3) + (2 \times 0.1470)}{0.1470 + 3}$$
$$= 1.7374 \text{ (4 d.p.)}$$

Now $f(1.7374) = -0.0182$ (4 d.p.)

There is a change of sign in the value of function $f(x)$ between $x = 1.7374$ and $x = 2$, so the root lies in the interval $1.7374 < x < 2$.
The table overleaf shows how the repeated use of the linear interpolation algorithm narrows the interval within which the root can lie. Recall that $f(1)$ is negative and $f(2)$ is positive.

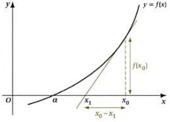
interval within which the root lies	approximate value of the root, x_n	$f(x_n)$
$2 < x < 2.5$	$x_1 = 2.1175$	$f(2.1175) = 0.0800$
$2.1775 < x < 2.5$	$x_2 = 2.1924$	$f(2.1924) = 0.0063$
$2.1924 < x < 2.5$	$x_3 = 2.1936$	$f(2.1936) = 0.0004$
$2.1936 < x < 2.5$		

Check this figure using the ZOOM and TRACE functions on a graphical calculator; remember to use radian mode to evaluate the sine ratio.

Since the value of $f(x)$ at $x = 2.1936$ is so close to zero, the root is probably very close to $x = 2.19$. In fact, $f(2.195) = -0.0066$, so there is a change of sign in the value of $f(x)$ between $x = 2.1936$ and $x = 2.195$. This means the root must lie in the interval $2.1936 < x < 2.195$, and can be taken to be 2.19 (2 d.p.). The graphs of $y = \sin x$ and $y = x^2 - 4$ therefore intersect at $x = 2.19$ (2 d.p.).

The Newton–Raphson method

The Newton–Raphson method is often simply called Newton's method. It usually rapidly converges to the root of an equation of the form $f(x) = 0$. It is quicker than both the interval bisection method and linear interpolation. Suppose $x = x_0$ is a first approximation to the root α of the equation $f(x) = 0$. A tangent drawn to the curve $y = f(x)$ at the point where $x = x_0$ can be extended to meet the x-axis at the point where $x = x_1$.



Spot the right-angled triangle formed by the tangent, the x-axis and the line $x = x_0$.

In most cases, x_1 will be a better approximation to the root α than x_0. By considering the gradient of this tangent, it follows that

$$f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

Now rearrange to make x_1 the subject of this equation.

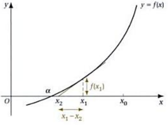
$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The process can be repeated, with a tangent drawn from the point on the curve where $x = x_1$. The tangent meets the x-axis at $x = x_2$. This is usually an even better approximation to the root α. By considering the gradient of this tangent, we find that

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

This process can be repeated as often as required.



In general, if x_n is a good approximation to the root of an equation of the form $f(x) = 0$, then a better approximation, x_{n+1}, can be found using the iterative formula

An iterative formula is one in which the output of one step of a calculation is used as input for the next step.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

◀ **Learn this important result.**

Here $f(x_n)$ and $f'(x_n)$ are the values of the function $f(x)$ and its derivative $f'(x)$ at $x = x_n$.
The Newton–Raphson method makes repeated use of this iterative formula until the required level of accuracy is achieved.

Example 7

Use Newton's method to find, correct to four decimal places, the root of the equation $e^x - x - 3 = 0$ that lies between $x = 1$ and $x = 2$.

Solution

In order to avoid rounding errors in the calculations, working should be done to at least two more decimal places than are required in the final answer. So in this example, working should be carried out to six decimal places.

Let $\quad f(x) = e^x - x - 3$

Then $\quad f'(x) = e^x - 1$

Take $x_0 = 1$ as a first approximation to the root.
Using Newton's method,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 1 - \frac{e^1 - 1 - 3}{e^1 - 1}$$
$$= 1.745\,930 \text{ (6 d.p.)}$$

Then $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 1.745\,930 - \frac{e^{1.745\,930} - 1.745\,930 - 3}{e^{1.745\,930} - 1}$$
$$= 1.537\,676 \text{ (6 d.p.)}$$

and $$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 1.537\,676 - \frac{e^{1.537\,676} - 1.537\,676 - 3}{e^{1.537\,676} - 1}$$
$$= 1.505\,904 \text{ (6 d.p.)}$$

Check that $x_4 = 1.505\,242$ (6 d.p.)
and that $x_5 = 1.505\,241$ (6 d.p.)

Notice that the value of x_n has only changed in the sixth decimal place during this last iteration. So the value of the root of the equation $e^x - x - 3 = 0$ must be $x = 1.5052$ (4 d.p.).
We could have taken $x_0 = 2$ as a first approximation to the root. Verify, using Newton's method, that the results of subsequent iterations are:

$x_1 = 1.626\,071$ $x_2 = 1.513\,974$ $x_3 = 1.505\,290$
$x_4 = 1.505\,241$ $x_5 = 1.505\,241$

All are obtained using six decimal places of working.

Again the conclusion is that the root of the equation is $x = 1.5052$ (4 d.p.).

It is possible to generate successive iterations quickly using the **last answer** (ANS) facility on a graphical calculator. For Newton's method, you would first enter the value of the first approximation x_0. The value of x_1 can then be found using the expression:

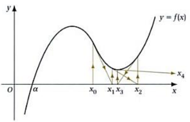
$$\text{ANS} - \frac{e^{\text{ANS}} - \text{ANS} - 3}{e^{\text{ANS}} - 1}$$

Using the **last entry** facility on the calculator, this expression can be entered again automatically, this time using the value of x_1 to produce x_2, and any further approximations that are required can be generated in the same way. Pressing the EXE or ENTER key gives the successive values $x_2, x_3, x_4, \ldots$ This method has the advantage of using the maximum number of decimal places available on the calculator in working out successive iterations of Newton's method.

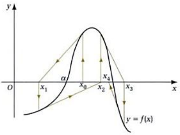
Notice that ANS replaces x_0 in the Newton–Raphson iterative formula.

We have solved the same equation in Examples 4 and 7. Using the interval bisection method it took five bisections to find the value of the root correct to just one decimal place. After five iterations, Newton's method produced a value correct to at least four decimal places. This illustrates how quickly the Newton–Raphson method usually converges to the root of an equation.

Occasionally, the Newton–Raphson method 'fails' and instead produces a series of values $x_0, x_1, x_2, \ldots$ that either diverge, or converge to a different root of the equation to that expected. These two different types of behaviour are illustrated in the following diagrams.



Notice how the values $x_0, x_1, x_2, \ldots$ diverge away from α.



Notice how the values $x_0, x_1, x_2, \ldots$ generated by the Newton–Raphson iterative formula converge towards a different root of the equation.

The main reason for these failures of the Newton–Raphson method is that the value of x_0 chosen as the first approximation is not close enough to the root for convergence to occur. Fortunately there are other iterative methods that can be used if a different choice of x_0 still proves unsuccessful.

Notice that the same iterative formula can be used to locate different roots *provided* that suitable starting points are used.

Fixed point iterative methods

It is usually possible to express an equation $f(x) = 0$ in the form $x = g(x)$. The root α of this type of equation can be found graphically by reading off the x-coordinate at the point of intersection of the $y = x$ and $y = g(x)$ graphs.

$$x_{n+1} = \tfrac{1}{4}(x_n^3 - 5)$$

$$x_{n+1} = \frac{5}{x_n^2 - 4}$$

$$x_{n+1} = \frac{4x_n + 5}{x_n^2}$$

However, $|g'(\alpha)| > 1$ in each of these cases, so the results of successive iterations starting with $x_0 = 2$ or $x_0 = 3$ do not converge. Instead they diverge. Check this using a graphical calculator.

13.1 Approximate Solutions of Equations

Exercise

Technique

1 a Show that the equation $x^3 - 3x - 5 = 0$ has a root in the interval $2 < x < 3$.

b Use linear interpolation to find a first approximation for this root and state the interval that contains the root. Give a reason for your answer.

2 a Find the consecutive integer values of x between which the equation $x^5 - 3x^2 + 10 = 0$ has a root.

b Use the interval bisection method to find the value of this root, correct to one decimal place.

c Taking a suitable value for the first approximation x_0, use the Newton–Raphson method to find the value of this root correct to four decimal places.

3 The sequence of values generated by the iterative formula $x_{n+1} = \sqrt[3]{9x_n + 11}$, with $x_0 = 3$, converges to a value α. Use this formula to find the value of α correct to three decimal places. State an equation satisfied by α. 3

4 a If $f(x) = xe^x - 5x - 6$, find an expression for $f'(x)$.

b Show that the equation $xe^x = 5x + 6$ has two roots. Show that one is in the interval $-2 < x < -1$. The other is in the interval $a < x < b$, where a and b are consecutive positive integers. Find a and b.

c Taking $x_0 = -2$ as an initial approximation, use Newton's method to find the value of the negative root. Write your answer correct to four decimal places.

d Use linear interpolation to find a first approximation for the value of the root that lies in the interval $a < x < b$. Give your answer to two decimal places.

5 a Show that the equation $2x^3 - 3x + 17 = 0$ has a root in the interval $-3 < x < -2$. 5 a

b Decide which two of the following iterative formulas could be used to find the root of this equation:

$$x_{n+1} = \frac{17}{2x_n^2 + 3} \qquad x_{n+1} = \frac{17}{3 - 2x_n^2}$$

$$x_{n+1} = \sqrt[3]{\tfrac{1}{2}(3x_n - 17)} \qquad x_{n+1} = \sqrt[3]{\frac{2x_n + 17}{3}}$$

c Taking $x_0 = -2$ as an initial approximation, determine which of these two iterative formulas converges towards a value for the root. Find this value correct to three decimal places.

Contextual

1 **a** Show that the equation $\ln x - x + 3 = 0$ has a root in the interval $4 < x < 5$.

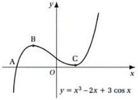
b Taking $x_0 = 4$ as an initial approximation:

i use the iterative formula $x_{n+1} = \ln(x_n) + 3$ to find the value of this root, correct to two decimal places

ii use the Newton–Raphson method to calculate its value, correct to four decimal places.

c Comment on the differences in the efficiency of these two methods.

2 The diagram shows the graph of $y = x^3 - 2x + 3\cos x$.



a Show that point A is located between $x = -2$ and $x = -1$.

b Using the Newton–Raphson method, calculate the x-coordinate of A, correct to three decimal places.

c Explain why the x-coordinates of B and C satisfy the equation $3x^2 - 2 - 3\sin x = 0$. By taking $x_0 = -1$ as a first approximation, use the Newton–Raphson method to solve this equation. Find the coordinates of B correct to three decimal places.

d Show that $3x^2 - 2 - 3\sin x = 0$ can also be solved numerically using the iterative formula $x_{n+1} = \sqrt{\frac{2}{3} + \sin x_n}$. Taking $x_0 = 1$ as a first approximation, find the coordinates of C correct to three decimal places.

3 The depth, D metres, of water in a harbour t hours after midday, is given by $D = 5 + \sin(0.48t) - 2\cos(0.48t)$.

a Show that when $D = 6.5$ metres, time t must satisfy the equation $\sin(0.48t) - 2\cos(0.48t) - 1.5 = 0$.

b Show that the water first reaches a depth of 6.5 metres between 3 and 4 o'clock in the afternoon.

c Use Newton's method to find this time to the nearest minute.

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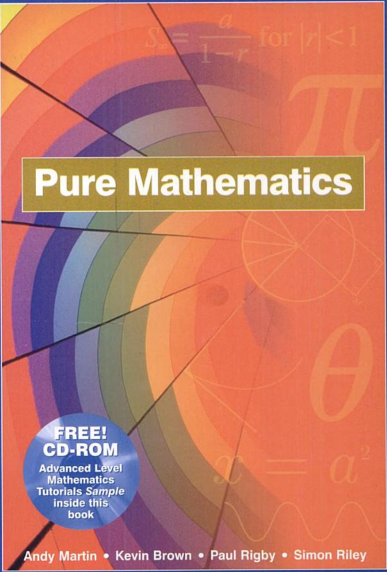

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